

THE BERGMAN KERNEL AND PROJECTION ON NON-SMOOTH WORM DOMAINS

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ABSTRACT. We study the Bergman kernel and projection on the worm domains

$$D_\beta = \left\{ \zeta \in \mathbb{C}^2 : \operatorname{Re}(\zeta_1 e^{-i \log |\zeta_2|^2}) > 0, |\log |\zeta_2|^2| < \beta - \frac{\pi}{2} \right\}$$

and

$$D'_\beta = \left\{ z \in \mathbb{C}^2 : |\operatorname{Im} z_1 - \log |z_2|^2| < \frac{\pi}{2}, |\log |z_2|^2| < \beta - \frac{\pi}{2} \right\}$$

for $\beta > \pi$. These two domains are biholomorphically equivalent via the mapping

$$D'_\beta \ni (z_1, z_2) \mapsto (e^{z_1}, z_2) \ni D_\beta.$$

We calculate the kernels explicitly, up to an error term that can be controlled.

As a result, we can determine the L^p -mapping properties of the Bergman projections on these worm domains. Denote by P the Bergman projection on D_β and by P' the one on D'_β . We calculate the sharp range of p for which the Bergman projection is bounded on L^p . More precisely we show that

$$P' : L^p(D'_\beta) \longrightarrow L^p(D'_\beta)$$

boundedly when $1 < p < \infty$, while

$$P : L^p(D_\beta) \longrightarrow L^p(D_\beta)$$

if and only if $2/(1 + \nu_\beta) < p < 2/(1 - \nu_\beta)$, where $\nu_\beta = \pi/(2\beta - \pi)$. Along the way, we give a new proof of the failure of Condition R on these worms.

Finally, we are able to show that the singularities of the Bergman kernel on the boundary are not contained in the boundary diagonal.

INTRODUCTION

The (smooth) worm domain was first created by Diederich and Fornæss [DF] to provide counterexamples to certain classical conjectures about the geometry of pseudoconvex domains. Chief among these examples is that the smooth worm is bounded and pseudoconvex and smooth yet its closure does not have a Stein neighborhood basis. More recently, thanks to work of Kiselman [Ki], Barrett [Ba2], and Christ [Chr], the worm has played an important role in the study of the $\bar{\partial}$ -Neumann

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problem and Condition R (see [CheS]). Recall that Condition R for a domain Ω is the assertion that the Bergman projection maps $C^\infty(\overline{\Omega})$ to $C^\infty(\overline{\Omega})$. Work of Bell [Be1] has shown that Condition R is closely related to the boundary regularity of biholomorphic mappings. See also [BoS2]. Condition R fails on the worm domains.

In more detail, in the seminal paper [Chr], Christ shows that the $\overline{\partial}$ -Neumann problem is not globally hypoelliptic on the smooth worm domain. It follows then that Condition R fails on the smooth worm. Christ's work provides considerable motivation for attaining a deeper and more detailed understanding of the Bergman kernel and projection on the worm domains. That is what the present paper achieves—for the particular worm domains¹

$$D_\beta = \left\{ \zeta \in \mathbb{C}^2 : \operatorname{Re}(\zeta_1 e^{-i \log |\zeta_2|^2}) > 0, |\log |\zeta_2|^2| < \beta - \frac{\pi}{2} \right\}$$

and

$$D'_\beta = \left\{ z \in \mathbb{C}^2 : |\operatorname{Im} z_1 - \log |z_2|^2| < \frac{\pi}{2}, |\log |z_2|^2| < \beta - \frac{\pi}{2} \right\}.$$

It should be noted that these two domains are biholomorphically equivalent via the mapping

$$(z_1, z_2) \ni D'_\beta \mapsto (e^{z_1}, z_2) \ni D_\beta.$$

Moreover, these domains are *not* smoothly bounded. Each boundary is only Lipschitz, and, particularly, its boundary is Levi flat, and clearly each is unbounded.

These domains are rather different from the smooth worm \mathcal{W}_β (see [CheS] and the discussion below), which has all boundary points, except those on a singular annulus in the boundary, strongly pseudoconvex. However our worm domain D_β is actually a model for the smoothly bounded \mathcal{W}_β (see, for instance, [Ba2]), and it can be expected that phenomena that are true on D_β will in fact hold on \mathcal{W}_β as well. We will say more about this symbiotic relationship below. We intend to explore the other worms, particularly the smooth worm, in future papers.

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¹Throughout this paper, we use notation for the different worm domains that is recorded in [CheS]. This notation has become rather standard.

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1. STATEMENT OF THE MAIN RESULTS

Throughout this paper we work on worm domains in \mathbb{C}^2 .

If $r > 0$ then we let $I_r = \{x \in \mathbb{R} : -r \leq x \leq r\} = [-r, r]$. Let $\beta > \pi/2$ and let \mathcal{W} denote the domain

$$\mathcal{W} = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1 - e^{i \log |z_2|^2}|^2 < 1 - \eta(\log |z_2|^2) \right\},$$

where

- (i) $\eta \geq 0$, η is even, η is convex;
- (ii) $\eta^{-1}(0) = I_{\beta-\pi/2} \equiv [-\beta + \frac{\pi}{2}, \beta - \frac{\pi}{2}]$;
- (iii) there exists a number $a > 0$ such that $\eta(x) > 1$ if $x < -a$ or $x > a$;
- (iv) $\eta'(x) \neq 0$ if $\eta(x) = 1$.

We will also write $\mathcal{W} = \mathcal{W}_\beta$. Notice that the slices of \mathcal{W} for z_2 fixed are discs centered on the unit circle with centers that wind $(2\beta - \pi)/2\pi$ times about that circle as $|z_2|$ traverses the range of values for which $\eta(\log |z_2|^2) < 1$.

It turns out that \mathcal{W} is smoothly bounded and pseudoconvex (see [CheS]). It is the classical, smooth worm studied by Diederich and Fornæss (see [DF]). Moreover, its boundary is *strongly* pseudoconvex except at the boundary points $(0, e^{i \log |z_2|^2})$ for $|\log |z_2|^2| < \beta - \pi/2$ —see [DF] or [CheS]. Notice that these points constitute an annulus in $\partial \mathcal{W}_\beta$.

Starting from work by Kiselman [Ki], Barrett [Ba2] showed that the Bergman projection $P_\mathcal{W}$ of \mathcal{W} does not map the s -order Sobolev space $W^s(\mathcal{W})$ boundedly into itself when $s \geq \nu_\beta$, where

$$\nu_\beta = \frac{\pi}{2\beta - \pi} \tag{1}$$

is half the reciprocal of the number of windings of \mathcal{W} .

In a later paper [Chr], Christ showed that the $\bar{\partial}$ -Neumann problem is not globally hypoelliptic on \mathcal{W} , a fact that is equivalent to the property that $P_\mathcal{W} : C^\infty(\overline{\mathcal{W}}) \not\rightarrow C^\infty(\overline{\mathcal{W}})$. In other words, \mathcal{W} does not satisfy Condition R .

It is still an open question, of great interest, whether a biholomorphic mapping of \mathcal{W} onto another smoothly bounded pseudoconvex domain extends smoothly to a diffeomorphism of the boundaries. In this direction, it is worth mentioning that Chen [Che1] has shown that the automorphism group of \mathcal{W} reduces to the rotations

in the z_2 -variable; hence all biholomorphic *self-maps* of \mathcal{W} do extend smoothly to the boundary.

It is also noteworthy that Chen [Che1] and Ligocka [L] have independently showed that the Bergman kernel of \mathcal{W} cannot lie in $C^\infty(\overline{\mathcal{W}} \times \overline{\mathcal{W}} \setminus \Delta)$, where Δ is the boundary diagonal. In fact, in [Che1] it is shown that this phenomenon is a consequence of the presence of a complex variety in the boundary of \mathcal{W} .

There are other worm domains, which are neither bounded nor smooth, that are nonetheless of considerable interest. These are: the unbounded, non-smooth worm with half-plane slices

$$D_\beta = \left\{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 : \operatorname{Re}(\zeta_1 e^{-i \log |\zeta_2|^2}) > 0, |\log |\zeta_2|^2| < \beta - \frac{\pi}{2} \right\} \quad (2)$$

and the unbounded, non-smooth worm with strip slices

$$D'_\beta = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |\operatorname{Im} z_1 - \log |z_2|^2| < \frac{\pi}{2}, |\log |z_2|^2| < \beta - \frac{\pi}{2} \right\}. \quad (3)$$

These domains are biholomorphically equivalent via the mapping

$$\Phi : D'_\beta \ni (z_1, z_2) \mapsto (e^{z_1}, z_2) \equiv (\zeta_1, \zeta_2) \in D_\beta.$$

Kiselman introduced the domains D_β , D'_β in [Ki]. In his fundamental work [Ba2], Barrett showed that it is possible to obtain information on the Bergman kernel and projection on \mathcal{W} from corresponding information on the non-smooth worm D_β by using an exhaustion and limiting argument.

It is therefore of interest, in its own right and as a model for the smooth case, to study the behavior of the Bergman kernel and projection on the domain D_β and its biholomorphic copy D'_β . It is obvious from the transformation rule of the Bergman kernel (see [Kr1]) that it suffices to obtain the expression for the kernel in one of the two domains. However, the L^p -mapping properties of the Bergman projections of the two domains turn out to be substantially different (just because L^p spaces of holomorphic functions do not transform canonically under biholomorphic maps when $p \neq 2$).

In this paper we determine, in Theorems 1 and 2, the explicit expression for the Bergman kernels for D_β and D'_β —up to controllable error terms. Once these are available we study the L^p -mapping properties of the corresponding Bergman projections in Theorems 3 and 4.

More precisely we prove the following results.

Theorem 1. *Let c_0 be a positive fixed constant. Let χ_1 be a smooth cut-off function on the real line, supported on $\{x : |x| \leq 2c_0\}$, identically 1 for $|x| < c_0$. Set $\chi_2 = 1 - \chi_1$.*

Let $\beta > \pi$ and let ν_β be defined as in (1). Let h be fixed, with

$$\nu_\beta < h < \min(1, 2\nu_\beta). \quad (4)$$

Then there exist functions F_1, F_2, \dots, F_8 and $\tilde{F}_1, \tilde{F}_2, \dots, \tilde{F}_8$, holomorphic in z and anti-holomorphic in w , for $z = (z_1, z_2)$, $w = (w_1, w_2)$ varying in a neighborhood of D'_β , and having size $\mathcal{O}(|\operatorname{Re} z_1 - \operatorname{Re} w_1|)$, together with all their derivatives, for $z, w \in \overline{D'_\beta}$, as $|\operatorname{Re} z_1 - \operatorname{Re} w_1| \rightarrow +\infty$. Moreover, there exist functions $E, \tilde{E} \in \mathcal{C}^\infty(\overline{D'_\beta} \times \overline{D'_\beta})$ such that

$$D_{z_1}^\alpha D_{w_1}^\gamma E(z, w), D_{z_1}^\alpha D_{w_1}^\gamma \tilde{E}(z, w) = \mathcal{O}(|\operatorname{Re} z_1 - \operatorname{Re} w_1|^{|\alpha|+|\gamma|}),$$

as $|\operatorname{Re} z_1 - \operatorname{Re} w_1| \rightarrow +\infty$. (Here, for $\lambda \in \mathbb{C}$, D_λ denotes the partial derivative in λ or $\bar{\lambda}$.)

Then the following holds. Set

$$\begin{aligned} K_b(z, w) &= \frac{F_1(z, w)}{(i(z_1 - \bar{w}_1) + 2\beta)^2 (e^{(\beta-\pi/2)} - z_2 \bar{w}_2)^2} \\ &\quad + \frac{F_2(z, w)}{(i(z_1 - \bar{w}_1) + 2\beta)^2 (z_2 \bar{w}_2 - e^{-[i(z_1 - \bar{w}_1) + \pi]/2})^2} \\ &\quad + \frac{F_3(z, w)}{(e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2)^2 (e^{(\beta-\pi/2)} - z_2 \bar{w}_2)^2} \\ &\quad + \frac{F_4(z, w)}{(i(z_1 - \bar{w}_1) - 2\beta)^2 (e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2)^2} \\ &\quad + \frac{F_5(z, w)}{(i(z_1 - \bar{w}_1) - 2\beta)^2 (z_2 \bar{w}_2 - e^{-(\beta-\pi/2)})^2} \\ &\quad + \frac{F_6(z, w)}{(e^{-[i(z_1 - \bar{w}_1) + \pi]/2} - z_2 \bar{w}_2)^2 (e^{-(\beta-\pi/2)} - z_2 \bar{w}_2)^2} \\ &\quad + \frac{F_7(z, w)}{(i(z_1 - \bar{w}_1) + 2\beta)^2 (e^{(\beta-\pi/2)} - z_2 \bar{w}_2) (e^{-[i(z_1 - \bar{w}_1) + \pi]/2} - z_2 \bar{w}_2)} \\ &\quad + \frac{F_8(z, w)}{(i(z_1 - \bar{w}_1) - 2\beta)^2 (e^{-(\beta-\pi/2)} - z_2 \bar{w}_2) (e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2)} + E(z, w) \\ &\equiv K_1(z, w) + \dots + K_8(z, w) + E(z, w). \end{aligned} \tag{5}$$

Define $K_{\tilde{b}}$ by replacing F_1, \dots, F_8 and E by $\tilde{F}_1, \dots, \tilde{F}_8$ and \tilde{E} and thus K_1, \dots, K_8 by $\tilde{K}_1, \dots, \tilde{K}_8$ respectively in formula (5).

Then there exist functions ϕ_1, ϕ_2 entire in z and \bar{w} (that is, anti-holomorphic in w), which are of size $\mathcal{O}(|\operatorname{Re} z_1 - \operatorname{Re} w_1|)$, together with all their derivatives, uniformly in all closed strips $\{|\operatorname{Im} z_1| + |\operatorname{Im} w_1| \leq C\}$, such that the Bergman kernel $K_{D'_\beta}$ on D'_β admits the asymptotic expansion

$$\begin{aligned} K_{D'_\beta}(z, w) &= \chi_1(\operatorname{Re} z_1 - \operatorname{Re} w_1) K_b(z, w) + \chi_2(\operatorname{Re} z_1 - \operatorname{Re} w_1) \left\{ e^{-h \operatorname{sgn}(\operatorname{Re} z_1 - \operatorname{Re} w_1) \cdot (z_1 - \bar{w}_1)} K_{\tilde{b}}(z, w) \right. \\ &\quad \left. + e^{-\nu_\beta \operatorname{sgn}(\operatorname{Re} z_1 - \operatorname{Re} w_1) \cdot (z_1 - \bar{w}_1)} \left(\frac{\phi_1(z_1, w_1)}{(e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2)^2} + \frac{\phi_2(z, w)}{(e^{-[i(z_1 - \bar{w}_1) + \pi]/2} - z_2 \bar{w}_2)^2} \right) \right\}. \end{aligned}$$

Here h is specified as in (4) above.

Theorem 2. *With the notation as in Theorem 1, there exist functions $g_1, g_2, G_1, G_2, \dots, G_8$ and $\tilde{G}_1, \tilde{G}_2, \dots, \tilde{G}_8$, holomorphic in ζ and anti-holomorphic in ω , for $\zeta = (\zeta_1, \zeta_2)$, $\omega = (\omega_1, \omega_2)$ varying in $\overline{D'_\beta} \setminus \{(0, z_2)\}$, such that*

$$\partial_{\zeta_1}^\alpha \partial_{\omega_1}^\gamma G(\zeta, \omega) = \mathcal{O}(|\zeta_1|^{-|\alpha|} |\omega_1|^{-|\gamma|}) \quad \text{as } |\zeta_1|, |\omega_1| \rightarrow 0 ,$$

where G denotes any of the functions g_j, G_j, \tilde{G}_j . Moreover, there exist functions $E, \tilde{E} \in \mathcal{C}^\infty(\overline{D'_\beta} \setminus \{(0, z_2)\} \times \overline{D'_\beta} \setminus \{(0, z_2)\})$ such that

$$D_{\zeta_1}^\alpha D_{\omega_1}^\gamma E(\zeta, \omega), D_{\zeta_1}^\alpha D_{\omega_1}^\gamma \tilde{E}(\zeta, \omega) = \mathcal{O}(|\zeta_1|^{-|\alpha|} |\omega_1|^{-|\gamma|}) \quad \text{as } |\zeta_1|, |\omega_1| \rightarrow 0 .$$

(Here D_λ , for $\lambda \in \mathbb{C}$, D_λ denotes the partial derivative in λ or $\bar{\lambda}$.)

Then the following holds. Set

$$\begin{aligned} H_b(\zeta, w) &= \frac{G_1(\zeta, w)}{(i \log(\zeta_1/\bar{\omega}_1) + 2\beta)^2 (e^{(\beta-\pi/2)} - \zeta_2 \bar{\omega}_2)^2} \\ &\quad + \frac{G_2(\zeta, w)}{(i \log(\zeta_1/\bar{\omega}_1) + 2\beta)^2 ((\zeta_1/\bar{\omega}_1)^{-i/2} e^{-\pi/2} - \zeta_2 \bar{\omega}_2)^2} \\ &\quad + \frac{G_3(\zeta, w)}{((\zeta_1/\bar{\omega}_1)^{-i/2} e^{\pi/2} - \zeta_2 \bar{\omega}_2)^2 (e^{(\beta-\pi/2)} - \zeta_2 \bar{\omega}_2)^2} \\ &\quad + \frac{G_4(\zeta, w)}{(i \log(\zeta_1/\bar{\omega}_1) - 2\beta)^2 ((\zeta_1/\bar{\omega}_1)^{-i/2} e^{\pi/2} - \zeta_2 \bar{\omega}_2)^2} \\ &\quad + \frac{G_5(\zeta, w)}{(i \log(\zeta_1/\bar{\omega}_1) - 2\beta)^2 (e^{-(\beta-\pi/2)} - \zeta_2 \bar{\omega}_2)^2} \\ &\quad + \frac{G_6(\zeta, w)}{((\zeta_1/\bar{\omega}_1)^{-i/2} e^{-\pi/2} - \zeta_2 \bar{\omega}_2)^2 (e^{-(\beta-\pi/2)} - \zeta_2 \bar{\omega}_2)^2} \\ &\quad + \frac{G_7(\zeta, w)}{(i \log(\zeta_1/\bar{\omega}_1) + 2\beta)^2 (e^{(\beta-\pi/2)} - \zeta_2 \bar{\omega}_2) ((\zeta_1/\bar{\omega}_1)^{-i/2} e^{-\pi/2} - \zeta_2 \bar{\omega}_2)} \\ &\quad + \frac{G_8(\zeta, w)}{(i \log(\zeta_1/\bar{\omega}_1) - 2\beta)^2 (e^{-(\beta-\pi/2)} - \zeta_2 \bar{\omega}_2) ((\zeta_1/\bar{\omega}_1)^{-i/2} e^{\pi/2} - \zeta_2 \bar{\omega}_2)} + E(\zeta, w) \\ &\equiv H_1(\zeta, \omega) + \dots + H_8(\zeta, \omega) + E(\zeta, \omega) . \end{aligned} \tag{6}$$

Define H_b by replacing G_1, \dots, G_8 and E by $\tilde{G}_1, \dots, \tilde{G}_8$ and \tilde{E} , and H_1, \dots, H_8 by $\tilde{H}_1, \dots, \tilde{H}_8$, respectively.

Then, setting $t = |\zeta_1| - |\omega_1|$, we have this asymptotic expansion for the Bergman kernel on D_β :

$$\begin{aligned} & K_{D_\beta}((\zeta_1, \zeta_2), (\omega_1, \omega_2)) \\ &= \chi_1(t) \frac{H_b(\zeta, \omega)}{\zeta_1 \bar{\omega}_1} + \chi_2(t) \left\{ \left(\frac{|\zeta_1|}{|\omega_1|} \right)^{-h \operatorname{sgnt}} e^{-h \operatorname{sgnt} \cdot (\arg \zeta_1 + \arg \omega_1)} \frac{H_b(\zeta, \omega)}{\zeta_1 \bar{\omega}_1} \right. \\ & \quad + \left(\frac{|\zeta_1|}{|\omega_1|} \right)^{-\nu_b \operatorname{sgnt}} e^{-\nu_b \operatorname{sgnt} \cdot (\arg \zeta_1 + \arg \omega_1)} \left(\frac{g_1(\zeta_1, \omega_1)}{\zeta_1 \bar{\omega}_1} \cdot \frac{1}{((\zeta_1/\bar{\omega}_1)^{-i/2} e^{\pi/2} - z_2 \bar{\omega}_2)^2} \right. \\ & \quad \left. \left. + \frac{g_2(\zeta, \omega)}{\zeta_1 \bar{\omega}_1} \cdot \frac{1}{((\zeta_1/\bar{\omega}_1)^{-i/2} e^{-\pi/2} - z_2 \bar{\omega}_2)^2} \right) \right\}, \end{aligned}$$

where h is defined in (4).

Theorem 3. Let P denote the Bergman projection on the domain D_β , $\beta > \pi$. Then

$$P : L^p(D_\beta) \rightarrow L^p(D_\beta)$$

is bounded if and only if

$$\frac{2}{1 + \nu_\beta} < p < \frac{2}{1 - \nu_\beta}.$$

Theorem 4. Let P' denote the Bergman projection on the domain D'_β , $\beta > \pi$. Then

$$P' : L^p(D'_\beta) \rightarrow L^p(D'_\beta)$$

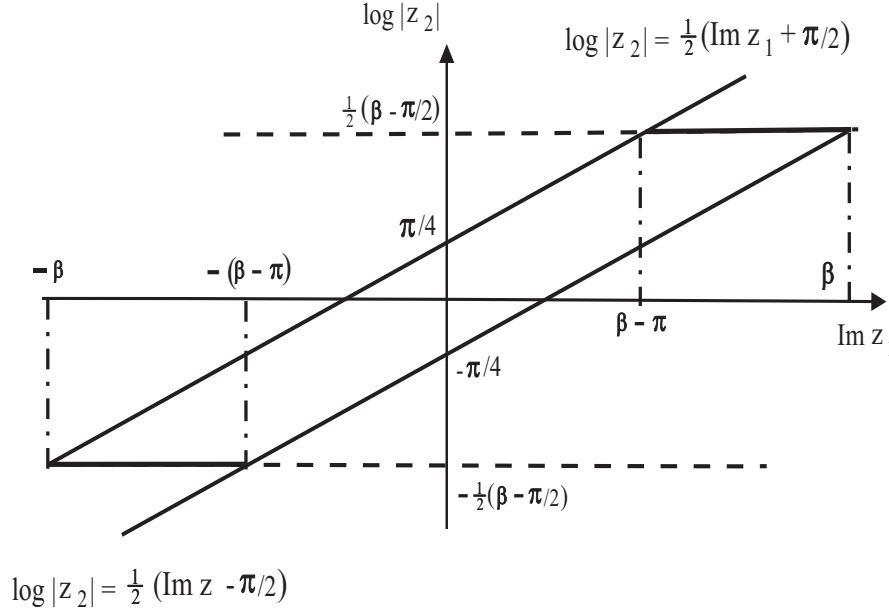
is bounded for $1 < p < \infty$.

Remark. A comment about the parameter β is now in order. The definition of D_β and D'_β (as well as the one of \mathcal{W}_β) requires that $\beta > \pi/2$.

However, for simplicity of the arguments, in this paper we restrict ourselves to the case $\beta > \pi$. This is not a serious constraint. The most interesting situations occur as $\beta \rightarrow +\infty$ (which means more twists in the geometry of the worm), and we believe that the expansion of the Bergman kernels for D_β and D'_β (Theorems 1 and 2) will extend to the case $\pi/2 < \beta \leq \pi$.

On the other hand, the result about the L^p boundedness of the Bergman projection on D_β (Theorem 4) requires that ν_b , defined in (1), satisfies $\nu_\beta < 1$. It is immediate to check that $\nu_\beta < 1$ if and only if $\beta > \pi$. It would therefore be of interest to study the L^p boundedness of the Bergman projection on D_β when $\pi/2 < \beta \leq \pi$.

The paper is organized as follows. In Section 2 we discuss the singularities of the kernels whose expansions are given in Theorems 1 and 2. Theorems 3 and 4 of that section treat the L^p boundedness of the Bergman projection. Section 2 also contains a comment concerning the failure of Condition R on the worm domains.

FIGURE 1. A representation of the domain D'_β .

In Sections 3 and 4, assuming the validity of the expansions in Theorems 1 and 2, we prove Theorems 4 and 3, respectively.

Section 4 records (Theorem 4.2) an explicit result about limitations of L^p boundedness of the Bergman projection on the smooth worm \mathcal{W}_β . This result is proved by way of an exhaustion procedure à la Barrett [Ba2].

Sections 5 through 11 are devoted to the proof of the expansions (i.e., Theorems 1 and 2) for the Bergman kernels on the worms D'_β and D_β . We do the bulk of our work in this paper directly with D'_β . At the end, we shall transfer the result to D_β via a biholomorphic mapping.

It is worth noting that the proofs of a number of our more technical results, including Proposition 9.1, Lemmas 9.2–9.5 and Lemma 11.1, are relegated to Section 11 of the paper.

2. ANALYSIS OF THE SINGULARITY OF THE BERGMAN KERNELS

Singularities of the Kernel on D'_β . We can draw a picture of the section of D'_β in the $(\text{Im } z_1, \log |z_2|)$ -plane.

Recall that

$$D'_\beta = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |\text{Im } z_1 - \log |z_2|^2| < \frac{\pi}{2}, |\log |z_2|^2| < \beta - \frac{\pi}{2} \right\},$$

and recall the expansion of $K_{D'_\beta}$ given in Theorem 1.

We begin by analyzing the behavior at the “bounded portion of the boundary”. Then (using the notation from Theorem 1) we notice the following facts:

(K_1). For $z, w \in D'_\beta$ the term K_1 becomes singular (if and) only if

$$z_2 \bar{w}_2 \rightarrow e^{\beta - \pi/2} \quad \text{or} \quad \text{Im}(z_1 - \bar{w}_1) \rightarrow 2\beta .$$

This can happen only if $\log |z_2|^2, \log |w_2|^2 \rightarrow \beta - \pi/2$. Thus K_1 is singular only when both z and w tend to the top side of the domain in Figure 1.

Notice that the top right corner of the domain in Figure 1 does not correspond to a single point in the domain D'_β . In fact, we could rotate in the z_2 -variable as well as translate the z_1 -variable by a real constant. Thus the singularity of the kernel K_1 is not contained in the boundary diagonal of $\overline{D'_\beta} \times \overline{D'_\beta}$, since for $w \in D'_\beta$ fixed, the singular set is $\{(z_1, z_2) \in \overline{D'_\beta} : |z_2| = e^{[\beta - \pi/2]/2}, -(\beta - \pi) < \text{Im } z_1 < \beta\}$. The interior of this set has real dimension 3. This phenomenon appears in the case of all the other terms, and we shall not repeat this comment again.

(K_5). This term is symmetric to K_1 and it is singular as

$$\log |z_2|^2, \log |w_2|^2 \rightarrow -(\beta - \pi/2) \quad \text{and, in this case, also when} \quad \text{Im } z_1, \text{Im } w_1 \rightarrow -\beta .$$

Thus, K_5 is singular only when both z and w tend to the lower side of the domain in Figure 1.

(K_2). For $z, w \in D'_\beta$ the term K_2 becomes singular (if and) only if

$$z_2 \bar{w}_2 \rightarrow e^{-[i(z_1 - \bar{w}_1) + \pi]/2} \quad \text{or} \quad \text{Im}(z_1 - \bar{w}_1) \rightarrow 2\beta .$$

This can happen only if $\log |z_2|^2 \rightarrow \text{Im } z_1 - \frac{\pi}{2}$ and $\log |z_2|^2 \rightarrow \text{Im } z_1 - \frac{\pi}{2}$ and thus K_2 is singular only when both z and w tend to the right oblique line that bounds the domain in Figure 1.

(K_4). This term is symmetric to K_2 and it is singular on the left oblique line of the domain in Figure 1.

(K_3, K_6). The kernel K_3 is singular when

$$z_2 \bar{w}_2 \rightarrow e^{[\pi - i(z_1 - \bar{w}_1)]/2} \quad \text{or} \quad z_2 \bar{w}_2 \rightarrow e^{\beta - \pi/2} .$$

Therefore K_3 is singular when both z and w tend to either the upper horizontal line, or the right oblique line on the boundary of the domain in Figure 1.

Analogously, the kernel K_6 is singular when both z and w tend to either the lower horizontal line, or the left oblique line on the boundary of the domain in Figure 1.

(K_7, K_8). Finally, these kernels have singularities of types that have been discussed already. In fact, there exists a constant $C > 0$ such that

$$|K_7(z, w)| \leq C(|K_1(z, w)| + |K_2(z, w)|),$$

and

$$|K_8(z, w)| \leq C(|K_4(z, w)| + |K_5(z, w)|),$$

We finish this part by noticing that, as $|\operatorname{Re} z_1|, |\operatorname{Re} w_1| \rightarrow +\infty$, the principal term of the kernel is $\approx e^{-\nu_\beta |\operatorname{Re} z_1 - \operatorname{Re} w_1|}$, as Theorem 1 clearly indicates.

Singularities of the Kernel on D_β . Now we turn to the Bergman kernel for the domain D_β ; we recall that the latter is given by

$$D_\beta = \left\{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 : \operatorname{Re}(\zeta_1 e^{-i \log |\zeta_2|^2}) > 0, |\log |\zeta_2|^2| < \beta - \frac{\pi}{2} \right\}.$$

The kernel is described in Theorem 2.

First of all, we comment on the definition of the function $\log \zeta_1$ on D_β . For $\zeta = (\zeta_1, \zeta_2) \in D_\beta$ and ζ_2 fixed, ζ_1 varies in the half-plane $\operatorname{Re}(\zeta_1 e^{-i \log |\zeta_2|^2}) > 0$, that is,

$$-\frac{\pi}{2} + \log |\zeta_2|^2 < \arg \zeta_1 < \frac{\pi}{2} + \log |\zeta_2|^2. \quad (7)$$

Therefore $\log \zeta_1$ is well defined as a function of $(\zeta_1, \zeta_2) \in D_\beta$.

Notice also that D_β can be described by inequality (7) together with the condition $-(\beta - \pi/2) < \log |\zeta_2|^2 < \beta - \pi/2$. These inequalities also imply that $-\beta < \arg \zeta_1 < \beta$.

The terms $i \log(\zeta_1/\bar{\omega}_1) \pm 2\beta$ in the expansion for the kernel become singular as $\arg \zeta_1, \arg \omega_1 \rightarrow \pm\beta$. But this can happen only if $\log |\zeta_2|^2, \log |\omega_2|^2 \rightarrow \pm(\beta - \pi/2)$, that is $|\zeta_2 \bar{\omega}_2| \rightarrow e^{\pm(\beta - \pi/2)}$.

Next observe that $|\zeta_2 \bar{\omega}_2| \rightarrow e^{\pi/2} |(\zeta_1/\bar{\omega}_1)^{\pm i/2}| = e^{\pi/2 \mp (\arg \zeta_1 + \arg \omega_1)/2}$, which occurs exactly when

$$\log |\zeta_2|^2 \rightarrow \frac{\pi}{2} \mp \arg \zeta_1 \quad \text{and} \quad \log |\omega_2|^2 \rightarrow \frac{\pi}{2} \mp \arg \omega_1.$$

Failure of Condition R. From the expansion given by Theorem 2 it is clear that, for any $\omega \in D_\beta$ fixed, the $A^2(D_\beta)$ -function $K_{D_\beta}(\cdot, \omega)$ does not belong to $L^2(D_\beta, |\zeta_1|^{-2\nu_\beta} dV)$. This in particular implies that $K_{D_\beta}(\cdot, \omega) \notin W^{\nu_\beta}(D_\beta)$ and that, therefore, condition R fails on D_β . This fact was already implicit on the work of Barrett [Ba2], and it appears explicitly in [CheS] Prop. 6.5.5.

3. BOUNDEDNESS OF THE BERGMAN PROJECTION ON D'_β

We now come to the proof of Theorem 4. By the asymptotic expansion of the Bergman kernel of D'_β given in Theorem 1 and the discussion in the previous section we have that

$$\begin{aligned}
& |K_{D'_\beta}(z, w)| \\
& \leq C e^{-h|\operatorname{Re} z_1 - \operatorname{Re} w_1|} \left(|K_1(z, w)| + \cdots + |K_6(z, w)| \right) \\
& \quad + C e^{-\nu_\beta |\operatorname{Re} z_1 - \operatorname{Re} w_1|} \left(\left| e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2 \right|^{-2} + \left| e^{-[i(z_1 - \bar{w}_1) + \pi]/2} - z_2 \bar{w}_2 \right|^{-2} \right) \\
& \equiv B_1(z, w) + \cdots + B_8(z, w), \tag{8}
\end{aligned}$$

where h is as in (4) of Theorem 1. Therefore, in order to prove the boundedness of the Bergman projection $P_{D'_\beta}$, it suffices to prove the boundedness of the operators T_{B_1}, \dots, T_{B_8} with positive kernels B_1, \dots, B_8 , respectively.

We will show that the operators $T_{B_1}, T_{B_2}, T_{B_3}$ and T_{B_7} and T_{B_8} are bounded on $L^p(D'_\beta)$, for $1 < p < \infty$. The boundedness of the remaining operators will follow by completely analogous arguments because of the symmetries discussed in the previous section.

In the course of the proof of Theorem 4 and of the next few lemmas, we will make use of the classical Schur's lemma (for which see, e.g., [Z], or [Sa]). We will also use the following estimates of "Forelli-Rudin" type (see [FR]).

Lemma 3.1. (i) *Let $0 < R < Q$, $0 < p < 1$ and $q > 0$. Let λ, τ denote (single) complex variables. Then there exists a constant $C > 0$ such that, for all $|\tau| < R$, we have*

$$\begin{aligned}
& \iint_{|\lambda| < R} \frac{1}{(Q^2 - |\lambda|^2)^q} \frac{1}{(R^2 - |\lambda|^2)^p} \frac{1}{|Q^2 - \tau \bar{\lambda}|^2} dV(\lambda) \\
& \leq C \min \left(\frac{1}{(Q^2 - R^2)^q} \frac{1}{(1 - |\tau|^2)^p}, \frac{1}{(1 - |\tau|^2)^{p+q}} \right).
\end{aligned}$$

(ii) *Moreover, if $a > 0$, then there exists a constant $C > 0$ such that*

$$\int_{-\infty}^{+\infty} \frac{e^{-h|x|}(1 + |x|)}{x^2 + a^2} dx \leq C \cdot \frac{1}{a}$$

as $a \rightarrow 0^+$. Here h is a fixed positive constant.

Proof. Fact **(i)** is an immediate consequence of the classical Forelli-Rudin type estimates (e.g. see [Z]), since $(Q^2 - |\lambda|^2)^{-q} \leq (Q^2 - R^2)^{-q}$.

Fact **(ii)** follows by the change of variables $x = ay$ and a simple pointwise estimate. We leave the elementary details to the reader. \square

Proposition 3.2. For $z, w \in D'_\beta$ let

$$B_1(z, w) = \frac{e^{-h|\operatorname{Re} z_1 - \operatorname{Re} w_1|} (1 + |\operatorname{Re} z_1 - \operatorname{Re} w_1|)}{|i(z_1 - \bar{w}_1) + 2\beta|^2 |e^{\beta - \pi/2} - z_2 \bar{w}_2|^2},$$

where h is defined by (4), and let $T = T_{B_1}$ be the integral operator

$$Tf(z) \equiv T_{B_1}f(z) = \int_{D'_\beta} B_1(z, w)f(w)dV(w).$$

Then $T : L^p(D'_\beta) \rightarrow L^p(D'_\beta)$ for $1 < p < \infty$.

Proof. Let $a > 0$ be a number to be specified later and define

$$\varphi(w) = [(e^{\beta - \pi/2} - |w_2|^2)(\beta - \operatorname{Im} w_1)]^{-a}. \quad (9)$$

We wish to show (following the paradigm of Schur's lemma) that there exists a constant $C > 0$ such that

$$\int_{D'_\beta} B_1(z, w)\varphi^p(w)dV(w) \leq C\varphi^p(z) \quad (10)$$

and

$$\int_{D'_\beta} B_1(z, w)\varphi^{p'}(w)dV(w) \leq C\varphi^{p'}(z) \quad (11)$$

for all $z \in D'_\beta$, where p' is the exponent conjugate to p .

We write $w_1 = t + iu$, and we break the region of integration into three parts:

$$\begin{aligned} E_1 &= D'_\beta \cap \{w : -\beta < u < -\beta + \pi\}, \\ E_2 &= D'_\beta \cap \{w : -\beta + \pi < u < \beta - \pi\}, \\ E_3 &= D'_\beta \cap \{w : \beta - \pi < u < \beta\}. \end{aligned}$$

Let I_1 , I_2 , and I_3 respectively denote the integrals over E_1 , E_2 and E_3 . Then, applying Lemma 3.1, we see that

$$\begin{aligned} I_1 &= \int_{E_1} B_1(z, w)\varphi^p(w)dV(w) \\ &\leq \int_{-\infty}^{+\infty} \int_{-\beta}^{-\beta+\pi} \frac{e^{-h(|\operatorname{Re} z_1 - t|)} (1 + |\operatorname{Re} z_1 - t|)}{(\operatorname{Re} z_1 - t)^2 + (2\beta - \operatorname{Im} z_1 - u)^2} \frac{1}{(\beta - u)^{ap}} \\ &\quad \times \iint_{e^{[-\beta+\pi/2]/2} < |w_2| < e^{[u+\pi/2]/2}} \frac{1}{(e^{\beta - \pi/2} - |w_2|^2)^{ap} |e^{\beta - \pi/2} - z_2 \bar{w}_2|^2} dV(w_2) du dt \\ &\leq C \frac{1}{(e^{\beta - \pi/2} - |z_2|^2)^{ap}} \int_{-\infty}^{+\infty} \int_{-\beta}^{-\beta+\pi} \frac{e^{-h(|\operatorname{Re} z_1 - t|)} (1 + |\operatorname{Re} z_1 - t|)}{(\operatorname{Re} z_1 - t)^2 + (2\beta - \operatorname{Im} z_1 - u)^2} \frac{1}{(\beta - u)^{ap}} du dt \\ &\leq C \frac{1}{(e^{\beta - \pi/2} - |z_2|^2)^{ap}} \\ &\leq C\varphi^p(z). \end{aligned}$$

The estimate for I_2 is similar. Applying Lemma 3.1 (ii) we have

$$\begin{aligned}
I_2 &= \int_{E_2} B_1(z, w) \varphi^p(w) dV(w) \\
&\leq \int_{-\infty}^{+\infty} \int_{-\beta+\pi}^{\beta-\pi} \frac{e^{-h(|\operatorname{Re} z_1 - t|)} (1 + |\operatorname{Re} z_1 - t|)}{(\operatorname{Re} z_1 - t)^2 + (2\beta - \operatorname{Im} z_1 - u)^2} \frac{1}{(\beta - u)^{ap}} \\
&\quad \times \iint_{e^{[-u+\pi/2]/2} < |w_2| < e^{[u+\pi/2]/2}} \frac{1}{(e^{\beta-\pi/2} - |w_2|^2)^{ap} |e^{\beta-\pi/2} - z_2 \bar{w}_2|^2} dV(w_2) dudt \\
&\leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \int_{-\infty}^{+\infty} \int_{-\beta+\pi}^{\beta-\pi} \frac{e^{-h(|\operatorname{Re} z_1 - t|)} (1 + |\operatorname{Re} z_1 - t|)}{(\operatorname{Re} z_1 - t)^2 + (2\beta - \operatorname{Im} z_1 - u)^2} \frac{1}{(\beta - u)^{ap}} dudt \\
&\leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \\
&\leq C \varphi^p(z) .
\end{aligned}$$

Finally, we estimate I_3 . For $0 < a < 1/p$ we have

$$\begin{aligned}
I_3 &= \int_{E_3} B_1(z, w) \varphi^p(w) dV(w) \\
&\leq \int_{-\infty}^{+\infty} \int_{\beta-\pi}^{\beta} \frac{e^{-h(|\operatorname{Re} z_1 - t|)} (1 + |\operatorname{Re} z_1 - t|)}{(\operatorname{Re} z_1 - t)^2 + (2\beta - \operatorname{Im} z_1 - u)^2} \frac{1}{(\beta - u)^{ap}} \\
&\quad \times \iint_{e^{[-u+\pi/2]/2} < |w_2| < e^{[\beta-\pi/2]/2}} \frac{1}{(e^{\beta-\pi/2} - |w_2|^2)^{ap} |e^{\beta-\pi/2} - z_2 \bar{w}_2|^2} dV(w_2) dudt \\
&\leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \int_{-\infty}^{+\infty} \int_{\beta-\pi}^{\beta} \frac{e^{-h(|\operatorname{Re} z_1 - t|)} (1 + |\operatorname{Re} z_1 - t|)}{(\operatorname{Re} z_1 - t)^2 + (2\beta - \operatorname{Im} z_1 - u)^2} \frac{1}{(\beta - u)^{ap}} dudt \\
&\leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \int_{\beta-\pi}^{\beta} \frac{1}{2\beta - (\operatorname{Im} z_1 + u)} \frac{1}{(\beta - u)^{ap}} du . \tag{12}
\end{aligned}$$

Now, by a simple change of variables, we see that the last integral above equals

$$\begin{aligned}
\int_0^\pi \frac{1}{v + \beta - \operatorname{Im} z_1} \frac{1}{(1 - e^{-v})^{ap}} dv &\leq C \int_0^\pi \frac{1}{v + \beta - \operatorname{Im} z_1} \frac{1}{v^{ap}} dv \\
&\leq C \frac{1}{(\beta - \operatorname{Im} z_1)^{ap}} ,
\end{aligned}$$

again, as long as $0 < ap < 1$. Inserting this estimate into (12) we obtain that $I_3 \leq C \varphi^p(z)$, provided that $0 < a < 1/p$.

Now, we may repeat the previous argument, with p replaced by p' , to obtain that

$$\int_{D'_\beta} B_1(z, w) \varphi^{p'}(w) dV(w) \leq C \varphi^{p'}(z)$$

provided that $0 < a < 1/p'$.

Choosing a such that $0 < a < \min(1/p, 1/p')$ we can find φ so that (10) and (11) are satisfied. This concludes the proof. \square

Proposition 3.3. *For $z, w \in D'_\beta$ let*

$$B_2(z, w) = \frac{e^{-h|\operatorname{Re} z_1 - \operatorname{Re} w_1|} (1 + |\operatorname{Re} z_1 - \operatorname{Re} w_1|)}{|i(z_1 - \bar{w}_1) + 2\beta|^2 |e^{[-i(z_1 - \bar{w}_1) + \pi]/2} - z_2 \bar{w}_2|^2},$$

where h is as specified in (4), and let $T = T_{B_2}$ be the integral operator

$$Tf(z) = \int_{D'_\beta} B_2(z, w) f(w) dV(w).$$

Then $T : L^p(D'_\beta) \rightarrow L^p(D'_\beta)$ for $1 < p < \infty$.

Proof. The proof follows the same lines as the previous one.

For $0 < a < \min(1/p, 1/p')$ we now define

$$\varphi(w) = [(e^{[\operatorname{Im} w_1 + \pi/2]} - |w_2|^2)(\beta - \operatorname{Im} w_1)]^{-a} \quad (13)$$

and, again, we wish to show there exists a constant $C > 0$ such that

$$\int_{D'_\beta} B_2(z, w) \varphi^p(w) dV(w) \leq C \varphi^p(z)$$

and

$$\int_{D'_\beta} B_2(z, w) \varphi^{p'}(w) dV(w) \leq C \varphi^{p'}(z)$$

for all $z \in D'_\beta$.

We consider the former integral first. Once more we write $w_1 = t + iu$, break the region of integration into E_1 , E_2 and E_3 , defined as in the proof of Proposition 3.2, and call the respective integrals I_1 , I_2 and I_3 .

We begin with I_3 . Using Lemma 3.1, (i) and (ii), we have

$$\begin{aligned} I_3 &\leq C \int_{-\infty}^{+\infty} \int_{\beta-\pi}^{\beta} \frac{e^{-h(|\operatorname{Re} z_1 - t|)} (1 + |\operatorname{Re} z_1 - t|)}{(\operatorname{Re} z_1 - t)^2 + (2\beta - \operatorname{Im} z_1 - u)^2} \frac{1}{(\beta - u)^{ap}} \\ &\quad \times \iint_{e^{[-u+\pi/2]/2} < |w_2| < e^{[\beta-\pi/2]/2}} \frac{1}{(e^{[u+\pi/2]} - |w_2|^2)^{ap} |e^{\beta-\pi/2} - z_2 \bar{w}_2|^2} dV(w_2) du dt \\ &\leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \int_{-\infty}^{+\infty} \int_{\beta-\pi}^{\beta} \frac{e^{-h(|\operatorname{Re} z_1 - t|)} (1 + |\operatorname{Re} z_1 - t|)}{(\operatorname{Re} z_1 - t)^2 + (2\beta - \operatorname{Im} z_1 - u)^2} \frac{1}{(\beta - u)^{ap}} du dt \\ &\leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \int_{\beta-\pi}^{\beta} \frac{1}{(2\beta - \operatorname{Im} z_1 - u)(\beta - u)^{ap}} du \\ &\leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \cdot \frac{1}{(\beta - \operatorname{Im} z_1)^{ap}} \\ &\leq C \varphi^p(z). \end{aligned}$$

Next we turn to I_2 . We have that

$$\begin{aligned}
I_2 &\leq \int_{-\infty}^{+\infty} \int_{-\beta+\pi}^{\beta-\pi} \frac{e^{-h(|\operatorname{Re} z_1 - t|)} (1 + |\operatorname{Re} z_1 - t|)}{(\operatorname{Re} z_1 - t)^2 + (2\beta - \operatorname{Im} z_1 - u)^2} \frac{1}{(\beta - u)^{ap}} \\
&\quad \times \iint_{e^{[-u+\pi/2]/2} < |w_2| < e^{[u+\pi/2]/2}} \frac{1}{(e^{[u+\pi/2]} - |w_2|^2)^{ap} |e^{\beta-\pi/2} - z_2 \bar{w}_2|^2} dV(w_2) du dt \\
&\leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \int_{-\infty}^{+\infty} \int_{-\beta+\pi}^{\beta-\pi} \frac{e^{-h(|\operatorname{Re} z_1 - t|)} (1 + |\operatorname{Re} z_1 - t|)}{(\operatorname{Re} z_1 - t)^2 + (2\beta - \operatorname{Im} z_1 - u)^2} \frac{1}{(\beta - u)^{ap}} du dt \\
&\leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \\
&\leq C \varphi^p(z) .
\end{aligned}$$

Finally we estimate I_1 , where again we use the fact that the integrals in u and t are bounded uniformly in z_1 :

$$\begin{aligned}
I_1 &\leq C \int_{-\infty}^{+\infty} \int_{-\beta}^{-\beta+\pi} \frac{e^{-h(|\operatorname{Re} z_1 - t|)} (1 + |\operatorname{Re} z_1 - t|)}{(\operatorname{Re} z_1 - t)^2 + (2\beta - \operatorname{Im} z_1 - u)^2} \frac{1}{(\beta - u)^{ap}} \\
&\quad \times \iint_{e^{[-\beta+\pi/2]/2} < |w_2| < e^{[u+\pi/2]/2}} \frac{1}{(e^{[u+\pi/2]} - |w_2|^2)^{ap} |e^{\beta-\pi/2} - z_2 \bar{w}_2|^2} dV(w_2) du dt \\
&\leq C \frac{1}{(e^{\beta-\pi/2} - |z_1|^2)^{ap}} \\
&\leq C \varphi^p(z) .
\end{aligned}$$

Again, we may repeat the argument with p replaced by p' . This concludes the proof. \square

In the proof of the next proposition we shall again need the following ‘‘Forelli-Rudin type estimate’’ (see also Lemma 31. above).

Lemma 3.4. (i) *Let $0 < R < Q$, $\varrho > 0$ and let $m = \min(\varrho, Q)$. Let λ, τ denote (single) complex variables with $|\tau| < m$ and $|\lambda| < R$, and $\theta \in \mathbf{R}$. Then, for $0 < a, b < 1$, there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\iint_{|\lambda| < R} \frac{1}{|R\varrho - e^{i\theta}\tau\bar{\lambda}|^2} \cdot \frac{1}{|Q^2 - \tau\bar{\lambda}|^2} \cdot \frac{1}{(R^2 - |\lambda|^2)^a (Q^2 - |\lambda|^2)^b} dV(\lambda) \\
&\leq C \frac{1}{(Q^2 - R^2)^b} \cdot \frac{1}{(m^2 - |\tau|^2)^a} \cdot \frac{1}{|Q^2 - e^{i\theta}\frac{R}{\varrho}|\tau|^2|^2}
\end{aligned}$$

as $|\lambda| \rightarrow R^-$, $R \rightarrow S^-$, and $|\tau| \rightarrow \varrho^-$.

(ii) *Let $\delta > 0$ and $0 < b < 1$. Then there exists a constant $C > 0$ such that*

$$\int_{-\infty}^{+\infty} \frac{e^{-h|x|}(1 + |x|)}{|1 - be^{ix}|^{1+\delta}} \leq C \frac{1}{(1 - b)^\delta} .$$

Proof. Although this is a fairly straightforward proof, we sketch it here for completeness and for the reader's convenience.

In order to prove part (i) we claim that, for $0 < a < 1$, there exists a constant $C > 0$ such that, for all τ, τ' in the unit disk,

$$\begin{aligned} \iint_{|\lambda| < 1} \frac{1}{|1 - \tau' \bar{\lambda}|^2 |1 - z \bar{\lambda}|^2 (1 - |\lambda|^2)^a} dV(\lambda) \\ \leq C \frac{1}{|1 - \tau' \bar{\tau}|^2} \cdot \min\{(1 - |\tau'|^2)^{-a}, (1 - |\tau|^2)^{-a}\}. \end{aligned}$$

Given the claim, part (i) follows by noticing that $(Q^2 - |\lambda|^2)^{-b} \leq (Q^2 - R^2)^{-b}$ and rescaling.

In order to prove the claim, one first integrates over the region $E \equiv \{\lambda : |1 - \tau' \bar{\lambda}| \leq \gamma |1 - \tau \bar{\lambda}|\}$, for some $\gamma > 0$ to be chosen later. Then one integrates over the symmetric region $E' \equiv \{\lambda : |1 - \tau \bar{\lambda}| \leq \gamma |1 - \tau' \bar{\lambda}|\}$, and then on the unit disk with $E \cup E'$ removed.

The proof of (ii) is achieved by noticing that the integral is controlled by a constant times

$$\int_0^1 \frac{1}{(1-b)^{1+\delta} + x^{1+\delta}} dx \leq C \frac{1}{(1-b)^\delta}.$$

This concludes the (sketch of the) proof of the lemma. \square

Proposition 3.5. For $z, w \in D'_\beta$ let

$$B_3(z, w) = \frac{e^{-h|\operatorname{Re} z_1 - \operatorname{Re} w_1|} (1 + |\operatorname{Re} z_1 - \operatorname{Re} w_1|)}{|e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2|^2 |e^{(\beta - \pi/2)} - z_2 \bar{w}_2|^2},$$

where h is defined in (4), and let $T = T_{B_3}$ be the integral operator

$$T_{B_3} f(z) = \int_{D'_\beta} B_3(z, w) f(w) dV(w).$$

Then $T : L^p(D'_\beta) \rightarrow L^p(D'_\beta)$ for $1 < p < \infty$.

Proof. For $0 < a < \min(1/p, 1\pi')$ we now define

$$\varphi(w) = [(m(w)^2 - |w_2|^2)(e^{\beta - \pi/2} - |w_2|^2)]^{-a} \quad (14)$$

where $m(w) \equiv \min(e^{[\beta - \pi/2]/2}, e^{[\operatorname{Im} w_1 + \pi/2]/2})$. Again, we wish to show there exists a constant $C > 0$ such that

$$\int_{D'_\beta} B_3(z, w) \varphi^p(w) dV(w) \leq C \varphi^p(z)$$

and

$$\int_{D'_\beta} B_3(z, w) \varphi^{p'}(w) dV(w) \leq C \varphi^{p'}(z)$$

for all $z \in D'_\beta$.

Once more we write $w_1 = t + iu$, break the region of integration into E_1 , E_2 and E_3 , and call the respective integrals I_1 , I_2 and I_3 .

Then we have,

$$I_1 \leq C \int_{-\infty}^{+\infty} \int_{-\beta}^{-\beta+\pi} \iint_{e^{[-\beta+\pi/2]/2} < |w_2| < e^{[u+\pi/2]/2}} \frac{e^{-h(|\operatorname{Re} z_1 - t|)} (1 + |\operatorname{Re} z_1 - t|)}{|e^{[\pi + \operatorname{Im} z_1 + u - i(\operatorname{Re} z_1 - t)]/2} - z_2 \bar{w}_2|^2} \\ \times \frac{1}{|e^{\beta-\pi/2} - z_2 \bar{w}_2|^2} \cdot \frac{1}{(e^{[u+\pi/2]} - |w_2|^2)^{ap} (e^{\beta-\pi/2} - |w_2|^2)^{ap}} dV(w_2) dudt .$$

Now we change variables in the outer integral, setting $s = t - \operatorname{Re} z_1$, and we apply Lemma 3.4 (i) with $e^{[u+\pi/2]/2} = R$, $e^{[\beta-\pi/2]/2} = Q$, $e^{[\operatorname{Im} z_1 + \pi/2]/2} = \varrho$:

$$I_1 \leq \int_{-\infty}^{+\infty} \int_{-\beta}^{-\beta+\pi} \iint_{e^{[-\beta+\pi/2]/2} < |w_2| < e^{[u+\pi/2]/2}} \frac{e^{-h(|s|)} (1 + |s|)}{|e^{[\pi + \operatorname{Im} z_1 + u]/2} - e^{-is/2} z_2 \bar{w}_2|^2} \\ \times \frac{1}{|e^{\beta-\pi/2} - z_2 \bar{w}_2|^2} \cdot \frac{1}{(e^{[u+\pi/2]} - |w_2|^2)^{ap} (e^{\beta-\pi/2} - |w_2|^2)^{ap}} dV(w_2) duds \\ \leq C \frac{1}{(e^{[\operatorname{Im} z_1 + \pi/2]} - |z_2|^2)^{ap}} \int_{-\infty}^{+\infty} \int_{-\beta}^{-\beta+\pi} \frac{e^{-h(|s|)} (1 + |s|)}{|e^{\beta-\pi/2} - e^{-is/2} e^{[u-\operatorname{Im} z_1]/2} |z_2|^2|^2} \\ \times \frac{1}{(e^{\beta-\pi/2} - e^{[u+\pi/2]})^{ap}} duds \\ \leq C \frac{1}{(e^{[\operatorname{Im} z_1 + \pi/2]} - |z_2|^2)^{ap}} \\ \times \int_{-\beta}^{-\beta+\pi} \frac{1}{(e^{\beta-\pi/2} e^{[\operatorname{Im} z_1 + \pi/2]/2} - e^{[u+\pi/2]/2} |z_2|^2) (e^{\beta-\pi/2} - e^{[u+\pi/2]})^{ap}} du .$$

Here we have applied Lemma 3.4 (ii) to the outer integral in ds . Now it is easy to see that the integral in du is uniformly bounded so that

$$I_1 \leq C \frac{1}{(e^{[\operatorname{Im} z_1 + \pi/2]} - |z_2|^2)^{ap}} \leq C \varphi^p(z) .$$

The proof of the estimate for I_2 is identical to the one for I_1 and we skip the details.

Finally, for I_3 we begin by arguing as in the previous cases to obtain that

$$I_3 \leq \int_{-\infty}^{+\infty} e^{-h(|s|)} (1 + |s|) \int_{\beta-\pi}^{\beta} \iint_{e^{[-\beta+\pi/2]/2} < |w_2| < e^{[\beta-\pi/2]/2}} \frac{1}{|e^{[\pi + \operatorname{Im} z_1 + u]/2} - e^{-is/2} z_2 \bar{w}_2|^2} \\ \times \frac{1}{|e^{\beta-\pi/2} - z_2 \bar{w}_2|^2} \cdot \frac{1}{(e^{[u+\pi/2]} - |w_2|^2)^{ap} (e^{\beta-\pi/2} - |w_2|^2)^{ap}} dV(w_2) duds .$$

Again we apply first Lemma 3.4 (i), this time with $e^{[\beta-\pi/2]/2} = R$, $e^{[u+\pi/2]/2} = Q$, and $e^{[\operatorname{Im} z_1 + \pi/2]/2} = \varrho$, and then part (ii) of the same lemma. We obtain

$$\begin{aligned} I_3 &\leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \int_{-\infty}^{+\infty} \int_{\beta-\pi}^{\beta} \frac{e^{-h(|s|)}(1+|s|)}{|e^{[u+\pi/2]} - e^{-is/2} e^{-[\operatorname{Im} z_1 + \pi/2]/2} e^{(\beta-\pi/2)/2} |z_2|^2|^2} \\ &\quad \times \frac{1}{(e^{\beta-\pi/2} - e^{[u+\pi/2]})^{ap}} du ds \\ &\leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \\ &\quad \times \int_{\beta-\pi}^{\beta} \frac{1}{(e^{[u+\pi/2]} - e^{-[\operatorname{Im} z_1 + \pi/2]/2} e^{(\beta-\pi/2)/2} |z_2|^2)(e^{[u+\pi/2]} - e^{\beta-\pi/2})^{ap}} du . \end{aligned}$$

Now we recall that, for $y > 0$, if $0 < ap < 1$,

$$\int_0^{+\infty} \frac{1}{x^{ap}} \cdot \frac{1}{x+y} dy \leq C \frac{1}{y^{ap}}$$

as $y \rightarrow 0^+$. Then the last integral in the display above can be easily seen to be less or equal to a constant times $(e^{[\operatorname{Im} z_1 + \pi/2]/2} e^{(\beta-\pi/2)/2} - |z_2|^2)^{-ap}$. Therefore it follows that

$$I_3 \leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \cdot \frac{1}{(e^{[\operatorname{Im} z_1 + \pi/2]/2} e^{(\beta-\pi/2)/2} - |z_2|^2)^{ap}} \leq C \varphi^p(z) ,$$

as it is easy to check.

Again, we may repeat the argument with p replaced by p' and then obtain that $T : L^p(D'_\beta) \rightarrow L^p(D'_\beta)$ is bounded for $1 < p < \infty$. \square

End of the Proof of Theorem 4. In order to conclude the proof of Theorem 4 we only need to show that the operators T_{B_7} and T_{B_8} are bounded on $L^p(D'_\beta)$, for $1 < p < \infty$.

The estimates for these kernels are simpler than the ones in 3.2–3.5. In these last two cases it suffices to consider test functions φ equal to

$$(e^{[\operatorname{Im} w_1 + \pi/2]} - |w_2|^2)^{-a} \quad \text{and} \quad (|w_2|^2 - e^{[\operatorname{Im} w_1 - \pi/2]})^{-a}$$

respectively.

To avoid further repetitions, we shall leave the simple details to the reader. \square

4. BOUNDEDNESS OF THE BERGMAN PROJECTION ON D_β

We now turn our attention to the L^p boundedness of the Bergman projection operator on D_β . Even though the domain D_β is biholomorphic to D'_β , the L^p behavior of the Bergman projection on these two domains is not *a priori* identical. That is because the space of L^p holomorphic functions for $p \neq 2$ does not transform

canonically under the biholomorphic mapping. Put in other words, the Jacobian of the mapping plays a non-obvious role in the calculations. And in fact it turns out that (in contrast to the situation on D'_β) the projection P_{D_β} is bounded on $L^p(D_\beta)$ only for the restricted range $2/(1 + \nu_\beta) < p < 2/(1 - \nu_\beta)$. We shall now establish this last assertion.

We shall use our asymptotic expansion for the Bergman kernel to determine the L^p boundedness of the Bergman projection on the domain D_β . We remark that, in his review [Str] of Barrett's paper [Ba2], E. Straube describes some results that are related to those that are presented here.

We begin with the negative result. Let $1 < p < \infty$ and assume that $P : L^p(D_\beta) \rightarrow L^p(D_\beta)$ is bounded. It follows that for any $\zeta \in D_\beta$ fixed, $K_{D_\beta}(\cdot, \zeta) \in L^{p'}(D_\beta)$, where $p' = p/[p - 1]$ is the exponent conjugate to p .

For, if $P = P_{D_\beta}$ is bounded, then for all $f \in L^p(D_\beta)$ and all $\zeta \in D_\beta$,

$$|\langle f, K_{D_\beta}(\cdot, \zeta) \rangle| = |Pf(\zeta)| \leq c_\zeta \|Pf\|_{L^p} \leq C \|f\|_{L^p}.$$

Lemma 4.1. *For any $\zeta \in D_\beta$ it holds that $K_{D_\beta}(\cdot, \zeta) \in L^p(D_\beta)$ only if $2/(1 + \nu_\beta) < p < 2/(1 - \nu_\beta)$.*

Proof. Fix $\zeta \in D_\beta$ and define

$$\Omega_\zeta = \{\omega \in D_\beta : |\omega_1| < |\zeta_1|, \ 1/4 \leq |e^{\pi/2}(\zeta_1/\bar{\omega}_1)^{\pm i/2} - \zeta_2\bar{\omega}_2| \leq 1/2\}.$$

Recall the expansion for the kernel $K_{D_\beta}(\zeta, \omega)$ given in Theorem 2. Then, for $\omega \in \Omega_\zeta$, we have that

$$|H_b(\zeta, \omega)|, |H_{\bar{b}}(\zeta, \omega)| \leq C_\zeta$$

for some constant independent of ω , so that

$$|K_{D_\beta}(\zeta, \omega)| \geq c_\zeta |\omega_1|^{\nu_\beta - 1} \tag{15}$$

for $\omega \in \Omega_\zeta$.

Therefore

$$\begin{aligned} \int_{D_\beta} |K_{D_\beta}(\cdot, \zeta)|^{p'} dV(\omega) &\geq \int_{\Omega_\zeta} |K_{D_\beta}(\cdot, \zeta)|^{p'} dV(\omega) \\ &\geq c_\zeta \int_{\Omega_\zeta} (|\omega_1|^{\nu_\beta - 1})^{p'} dV(\omega) = c \int_0^{|\zeta_1|} r^{p'(\nu_\beta - 1) + 1} dr. \end{aligned}$$

Obviously for convergence we need $p'(\nu_\beta - 1) + 1 > -1$, that is $p' < 2/[1 - \nu_\beta]$. Hence if $p \geq 2/[1 - \nu_\beta]$ then the integral diverges. The other result, for $p \leq 2/[1 + \nu_\beta]$, follows by duality. This proves the lemma. \square

For the record, we record now the fact that we can use Barrett's exhaustion procedure (see [Ba2]) to obtain a negative result with the same indices on the smooth worm \mathcal{W} . Details of the proof appear in [KrP].

Theorem 4.2. *Let \mathcal{P} denote the Bergman projection on the smooth, bounded worm $\mathcal{W} = \mathcal{W}_\beta$, with $\beta > \pi/2$. Then, if $\mathcal{P} : L^p(\mathcal{W}_\beta) \rightarrow L^p(\mathcal{W}_\beta)$ is bounded, necessarily $2/[1 + \nu_\beta] < p < 2/[1 - \nu_\beta]$.*

Next we prove the positive part of Theorem 3.

Theorem 4.3. *Let $2/[1 + \nu_\beta] < p < 2/[1 - \nu_\beta]$. Then*

$$P_{D_\beta} : L^p(D_\beta) \rightarrow L^p(D_\beta)$$

is bounded.

In order to prove this fact we use the biholomorphism $\Phi : D'_\beta \rightarrow D_\beta$ given by $\Phi(z_1, z_2) = (e^{z_1}, z_2)$, so that $\det \text{Jac}_\mathbb{C} \Phi = e^{z_1}$.

Therefore, by the transformation rule for the Bergman kernels *via* biholomorphic mappings, we wish to show that the integral operator T with kernel $L(z, w) \equiv e^{-z_1} K_{D'_\beta}(z, w) e^{\overline{w}_1}$ is bounded on $L^p(D'_\beta)$ when $2/[1 + \nu_\beta] < p < 2/[1 - \nu_\beta]$.

As in the proof of Theorem 4, we are going to use Schur's lemma. With the notation of Section 3, recalling (8), we have that

$$\begin{aligned} |L(z, w)| &\leq C e^{\text{Re } w_1 - \text{Re } z_1} |K_{D'_\beta}(z, w)| \leq C e^{\text{Re } w_1 - \text{Re } z_1} (B_1(z, w) + \cdots B_8(z, w)) \\ &\equiv A_1(z, w) + \cdots A_8(z, w) . \end{aligned} \quad (16)$$

For each $j = 1, \dots, 8$, we wish to determine a positive function ψ on D'_β such that

$$\int_{D'_\beta} A_j(z, w) \psi^p(w) dV(w) \leq C \psi^p(z) .$$

As in the proof of Theorem 4, by symmetry it suffices to consider the cases of $j = 1, 2, 3$ and $j = 7$.

Proposition 4.4. *For $z, w \in D'_\beta$ let*

$$A_1(z, w) = \frac{e^{-h|\text{Re } z_1 - \text{Re } w_1|} e^{\text{Re } w_1 - \text{Re } z_1} (1 + |\text{Re } z_1 - \text{Re } w_1|)}{|i(z_1 - \overline{w}_1) + 2\beta|^2 |e^{\beta - \pi/2} - z_2 \overline{w}_2|^2} ,$$

for h as in (4), and let T be the integral operator

$$Tf(z) = \int_{D'_\beta} A_1(z, w) f(w) dV(w) .$$

Then $T : L^p(D'_\beta) \rightarrow L^p(D'_\beta)$ for $2/[1 + \nu_\beta] < p < 2/[1 - \nu_\beta]$.

Proof. Let $a, b > 0$ and define

$$\psi(w) = e^{-b \text{Re } w_1} [(e^{\beta - \pi/2} - |w_2|^2)(\beta - \text{Im } w_1)]^{-a} . \quad (17)$$

Arguing as in the proof of Proposition 3.2, dividing the region of integration into $E_1 \cup E_2 \cup E_3$ and performing the integrations in w_2 and $\text{Im } w_1$ (having set $t = \text{Re } w_1$),

we find that

$$\begin{aligned}
& \int_{E_1 \cup E_2} A_1(z, w) \psi^p(w) dV(w) \\
& \leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \int_{-\infty}^{+\infty} e^{-h(|\operatorname{Re} z_1 - t|)} e^{t - \operatorname{Re} z_1} (1 + |\operatorname{Re} z_1 - t|) e^{-bpt} dt \\
& = C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \int_{-\infty}^{+\infty} e^{-h(|s|)} e^s (1 + |s|) e^{-bp(s + \operatorname{Re} z_1)} ds \\
& = C \psi^p(z)
\end{aligned}$$

as long as $0 < 1 - bp < h$, that is $\frac{1}{p} - h < b < \frac{1}{p}$.

The same condition must be valid for the conjugate exponent p' , that is $\frac{1}{p'} - h < b < \frac{1}{p'}$, which translates into $1 - \frac{1}{p} - h < b < 1 - \frac{1}{p}$. Assume that $p > 2$ for the moment. Then the intervals $(\frac{1}{p} - h, \frac{1}{p})$ and $(1 - \frac{1}{p} - h, 1 - \frac{1}{p})$ have non-empty intersection if and only if

$$\frac{1}{p} > 1 - \frac{1}{p} - h;$$

that is if and only if $h > 1 - \frac{2}{p}$.

We recall here that h is assumed to satisfy $\nu_b < h < \min(1, 2\nu_b)$ (see Theorems 1, 2), and that p is momentarily assumed to satisfy $2 < p < 2/[1 - \nu_b]$. Then $1 - \frac{2}{p} < \nu_b$ and we are done.

The argument for $p < 2$ is similar, and we skip the details.

Next, arguing again as in the proof of Proposition 3.2, we see that

$$\begin{aligned}
& \int_{E_3} A_1(z, w) \psi^p(w) dV(w) \\
& \leq C \frac{1}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \int_{-\infty}^{+\infty} \int_{\beta-\pi}^{\beta} \frac{e^{-h(|\operatorname{Re} z_1 - t|)} e^{t - \operatorname{Re} z_1} (1 + |\operatorname{Re} z_1 - t|) e^{-bpt}}{[(\operatorname{Re} z_1 - t)^2 + (2\beta - \operatorname{Im} z_1 - u)^2] (\beta - u)^{ap}} du dt \\
& = C \frac{e^{-bp \operatorname{Re} z_1}}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \int_{-\infty}^{+\infty} \int_{\beta-\pi}^{\beta} \frac{e^{-h(|s|)} e^s (1 + |s|) e^{-bp(s + \operatorname{Re} z_1)}}{[s^2 + (2\beta - \operatorname{Im} z_1 - u)^2] (\beta - u)^{ap}} du ds \\
& \leq C \frac{e^{-bp \operatorname{Re} z_1}}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \int_{\beta-\pi}^{\beta} \frac{1}{2\beta - (\operatorname{Im} z_1 + u)} \frac{1}{(\beta - u)^{ap}} du \\
& \leq C \frac{e^{-bp \operatorname{Re} z_1}}{(e^{\beta-\pi/2} - |z_2|^2)^{ap}} \frac{1}{(\beta - \operatorname{Im} z_1)^{ap}} = C \psi^p(z).
\end{aligned}$$

This concludes the proof of the proposition. \square

The fact that the operators with integral kernels A_2 and A_3 are bounded on $L^p(D_\beta)$ for $2/[1 + \nu_\beta] < p < 2/[1 - \nu_\beta]$ is proved following the same pattern as in the proof of 3.2-3.5 and 4.4. To avoid repetitions we leave the details to the reader.

End of the Proof of Theorem 3. In order to finish the proof of Theorem 3, we show that the integral operator with kernel $A_7(z, w)$ is bounded on $L^p(D_\beta)$ for $2/[1+\nu_\beta] < p < 2/[1-\nu_\beta]$. Here

$$A_7(z, w) \equiv \frac{e^{\nu_\beta |\operatorname{Re} z_1 - \operatorname{Re} w_1|} (1 + |\operatorname{Re} z_1 - \operatorname{Re} w_1|) e^{\operatorname{Re} z_1 + \operatorname{Re} w_1}}{|e^{[-i(z_1 - \bar{w}_1) + \pi]/2} - z_2 \bar{w}_2|^2},$$

and we define

$$\psi(w) = e^{-b \operatorname{Re} w_1} (e^{[\operatorname{Im} w_1 + \pi/2]} - |w_2|^2)^{-a},$$

for some $a, b > 0$ to be selected later.

Then, arguing as in the previous proof, we see that, for $0 < a < 1/p$,

$$\begin{aligned} & \int_{D_\beta} A_7(z, w) \psi^p(w) dV(w) \\ & \leq C \frac{1}{(e^{[\operatorname{Im} z_1 + \pi/2]} - |z_2|^2)^{-ap}} \int_{-\infty}^{+\infty} e^{-\nu_\beta |\operatorname{Re} z_1 - t|} (1 + |\operatorname{Re} z_1 - t|) e^{\operatorname{Re} z_1 + t} e^{-bpt} dt \\ & \leq C \frac{e^{-bp \operatorname{Re} z_1}}{(e^{[\operatorname{Im} z_1 + \pi/2]} - |z_2|^2)^{-ap}} \int_{-\infty}^{+\infty} e^{-\nu_\beta |s|} (1 + |s|) e^{(1-bp)s} ds \\ & C \psi^p(z) \end{aligned}$$

as long as $0 < 1 - bp < \nu_b$.

The analogous condition with p' in place of p (and the same b) must hold. Again, we argue as in the proof of Proposition 4.4 to see that

$$b \in \left(1 - \frac{1}{p}, \frac{1}{p} - \nu_b\right) \quad \text{if } p < 2 \quad \text{and} \quad b \in \left(1 - \frac{1}{p} - \nu_b, \frac{1}{p}\right) \quad \text{if } p > 2$$

satisfies the required conditions.

This concludes the proof of Theorem 3. \square

5. DECOMPOSITION OF THE BERGMAN SPACE

In this section we begin our detailed analysis of the Bergman kernels on D_β and $D_{\beta'}$, leading to the asymptotic expansions in Theorem 1 and Theorem 2. We follow the calculations in [Ki] and [Ba2] in order to obtain a decomposition of the Bergman space on the domains D_β and D'_β . We in fact concentrate our attention on the latter domain. An analogous argument holds on D_β . Note that in the present context the usual transformation rule for the Bergman kernel will serve us well. Calculations on D'_β will transfer to D_β automatically.

Let the Bergman space $A^2(D'_\beta)$ be the collection of holomorphic functions that are square integrable with respect to Lebesgue volume measure dV on D'_β . Following Kiselman [Ki] and Barrett [Ba2], we decompose the Bergman space as follows. Using

the rotational invariance in z_2 and elementary Fourier series, each $f \in A^2(D'_\beta)$ can be written as

$$f = \sum_{j=-\infty}^{\infty} f_j,$$

where each f_j is holomorphic and satisfies $f_j(z_1, e^{i\theta} z_2) = e^{ij\theta} f_j(z_1, z_2)$ for θ real. In fact, such an f_j is given by

$$f_j(z_1, z_2) = \frac{1}{2\pi} \int_0^{2\pi} f(z_1, e^{i\theta} z_2) e^{-ij\theta} d\theta.$$

Therefore

$$\mathcal{H} = \bigoplus_{j \in \mathbb{Z}} \mathcal{H}^j,$$

where

$$\mathcal{H}^j = \{f \in L^2 : f \text{ is holomorphic and } f(w_1, e^{i\theta} w_2) = e^{ij\theta} f(w_1, w_2)\}.$$

Notice that the function

$$g_j(z_1, z_2) \equiv \left[\frac{1}{2\pi} \int_0^{2\pi} f(z_1, e^{i\theta} z_2) e^{-ij\theta} d\theta \right] z_2^{-j}$$

is holomorphic on D'_β and depends only on $|z_2|$. Therefore it must be locally constant in z_2 . But, for all z_1 , the set $\{z_2 : (z_1, z_2) \in D'_\beta\}$ is connected, so that $g_j(z_1, z_2) \equiv h_j(z_1)$. Since $f_j(z_1, z_2) = h_j(z_1) z_2^j$ is holomorphic on D'_β , it is easy to see that h_j must be holomorphic on the strip $\{z_1 : |y| < \beta\}$, where $z_1 = x + iy$.

The following result is contained in [Ba2] Section 3. (Further details can also be found in [KrP].)

Lemma 5.1. *Let $\beta > \pi/2$ and $f_j \in \mathcal{H}^j$. Then there exists a function h_j of one complex variable, holomorphic in the strip $S_\beta = \{z_1 = x + iy, : |y| < \beta\}$, such that $f_j(z_1, z_2) = h_j(z_1) z_2^j$. Moreover, h_j is square-integrable on S_β with respect to the weight*

$$\lambda_j(y) = (\chi_{\pi/2} * [e^{(j+1)(\cdot)} \chi_{\beta-\pi/2}]) (y).$$

Here χ_s denotes the characteristic function of the interval $[-s, s]$ for $s \geq 0$.

If K is the Bergman kernel for $A^2(D'_\beta)$ and K_j the Bergman kernel for \mathcal{H}^j , then we may write

$$K = \sum_{j=-\infty}^{\infty} K_j.$$

Notice that, by rotational the invariance property of \mathcal{H}^j , with $z = (z_1, z_2)$ and $w = (w_1, w_2)$, we have that

$$K_j(z, w) = H_j(z_1, w_1) z_2^j \overline{w_2^j}.$$

Our job, then, is to calculate each H_j , and thereby each K_j . The first step of this calculation is already done in [Ba2], Section 2:

Proposition 5.2. *Let $\beta > \pi/2$. Then*

$$H_j(z_1, w_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(z_1 - \bar{w}_1)\xi} \xi \left(\xi - \frac{j+1}{2}\right)}{\sinh(\pi\xi) \sinh\left[(2\beta - \pi)\left(\xi - \frac{j+1}{2}\right)\right]} d\xi. \quad (18)$$

The papers [Ki] and [Ba2] calculate and analyze only the Bergman kernel for \mathcal{H}^{-1} (i.e., the Hilbert subspace with index $j = -1$). This is attractive to do because certain “resonances” cause cancellations that make the calculations tractable when $j = -1$. One of the main thrusts of the present work is to perform the more difficult calculations for all j .

Therefore it follows that the Bergman kernel K' for D'_β can be written as

$$\begin{aligned} K'(z, w) &= \sum_{j \in \mathbb{Z}} H_j(z_1, w_1) (z_2 \bar{w}_2)^j \\ &= \sum_{j \in \mathbb{Z}} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(z_1 - \bar{w}_1)\xi} \xi \left(\xi - \frac{j+1}{2}\right)}{\sinh(\pi\xi) \sinh\left[(2\beta - \pi)\left(\xi - \frac{j+1}{2}\right)\right]} d\xi \right) (z_2 \bar{w}_2)^j. \end{aligned} \quad (19)$$

For $(z_1, z_2), (w_1, w_2) \in D'_\beta$ we set

$$\tau = z_1 - \bar{w}_1, \quad \lambda = z_2 \bar{w}_2.$$

Notice that, if z_1, w_1 vary in S_β , then τ varies in $S_{2\beta}$. Moreover, we set

$$g_j(\xi) = \frac{1}{2\pi} \frac{e^{i\tau\xi} \xi \left(\xi - \frac{j+1}{2}\right)}{\sinh(\pi\xi) \sinh\left[(2\beta - \pi)\left(\xi - \frac{j+1}{2}\right)\right]}$$

and

$$I_j(\tau) = \int_{-\infty}^{+\infty} g_j(\xi) d\xi. \quad (20)$$

Remark 5.3. Thus, in order to determine the expression of the Bergman kernel for D'_β , we have reduced ourselves to calculate

$$\sum_{j \in \mathbb{Z}} I_j(\tau) \lambda^j, \quad (21)$$

for $\tau \in S_{2\beta}$. The first task is to calculate $I_j(\tau)$ for each j . We shall distinguish two cases according to whether $|\operatorname{Re} \tau| > c_0$ or $|\operatorname{Re} \tau| \leq c_0$, for some fixed (small) constant c_0 .

When $|\operatorname{Re} \tau| > c_0$ we shall use the method of contour integrals, thus splitting I_j as sum of a residue R_j and of a term J_j coming from the contour integral. In this case, the expression of the Bergman kernel will be given by

$$\sum_{j \in \mathbb{Z}} R_j(\tau) \lambda^j + \sum_{j \in \mathbb{Z}} J_j(\tau) \lambda^j.$$

The term $\sum_{j \in \mathbb{Z}} R_j(\tau) \lambda^j$ coming from the sum of the residues contains the main singularity as $\operatorname{Re} \tau \rightarrow \pm\infty$.

When $|\operatorname{Re} \tau|$ remains bounded we shall compute the expression of the Bergman kernel by computing the sum $\sum_{j \in \mathbb{Z}} I_j(\tau) \lambda^j$ directly.

We begin with the analysis of the case $|\operatorname{Re}(\tau)| > c_0$, for some fixed constant $c_0 > 0$. The following result is elementary.

Lemma 5.4. *The function g_j is holomorphic in the plane except at the points*

$$\xi = ki, \quad k \in \mathbb{Z} \setminus \{0\}, \quad \xi = \frac{j+1}{2} + ik\nu_\beta, \quad k \in \mathbb{Z} \setminus \{0\},$$

where ν_β is defined in (1). Moreover, we have the residue

$$\operatorname{Res}_{\xi = \frac{j+1}{2} \pm i\nu_\beta} g_j = \frac{1}{2\pi i} \cdot \frac{\nu_\beta^2}{\pi} (\nu_\beta \mp i \frac{j+1}{2}) \frac{e^{\mp \nu_\beta \tau + i\tau(j+1)/2}}{\sinh(\pi \frac{j+1}{2} \pm i\nu_\beta \pi)}.$$

Proof. Since

$$\operatorname{Res}_{\xi = \frac{j+1}{2} \pm i\nu_\beta} \frac{1}{\sinh((2\beta - \pi)(\xi - \frac{j+1}{2}))} = -\frac{1}{2\beta - \pi},$$

the desired residues are

$$\pm \frac{1}{2\pi i} \cdot \frac{-i\nu_\beta}{2\beta - \pi} \cdot \frac{e^{i\tau\xi}}{\sinh(\pi\xi)} \Big|_{\xi = \frac{j+1}{2} \pm i\nu_\beta}.$$

The assertion follows at once. \square

Proposition 5.5. *Let $\beta > \pi$ and fix h such that*

$$\nu_\beta < h < \min(1, 2\nu_\beta).$$

For $\operatorname{Re} \tau \geq 0$ and $\operatorname{Re} \tau < 0$, define respectively

$$R_j(\tau) = 2\pi i \operatorname{Res}_{\xi = \frac{j+1}{2} \pm i\nu_\beta} g_j, \quad J_j(\tau) = \int_{-\infty}^{+\infty} g_j(\xi \pm ih) d\xi, \quad (22)$$

and recall that I_j is defined by (20). Then, for all $j \in \mathbb{Z}$,

$$I_j(\tau) = R_j(\tau) + J_j(\tau).$$

Proof. We assume that $\nu_\beta < 1$, i.e. $\beta > \pi$, let ν_β be as in (1), and fix h such that $\nu_\beta < h < \min(1, 2\nu_\beta)$ (that is, as in (4)).

According to whether $\operatorname{Re} \tau \geq 0$ or $\operatorname{Re} \tau < 0$, we choose the contour of integration γ_N^\pm to be a rectangular box, centered on the imaginary axis, with corners at $N + i0$,

$-N + i0$, $N \pm ih$, and $-N \pm ih$. Then

$$\begin{aligned} 2\pi i \operatorname{Res}_{\xi=\frac{j+1}{2} \pm i\nu_\beta} g_j &= \oint_{\gamma_N^\pm} g_j(\xi) d\xi \\ &= \int_{-N}^N g_j(\xi) d\xi + \oint_{I_N} g_j d\gamma + \oint_{I_{-N}} g_j d\gamma - \int_{-N}^N g_j(\xi \pm ih) d\xi. \end{aligned}$$

Here I_N and I_{-N} are the left and right edges of the box. Of course the first and last integrals on the right are taken over the (long) top and (long) bottom of the rectangles. The other two (i.e., second and third) integrals on the right are over the (short, vertical) ends of the rectangles. We next verify that the latter two short integrals tend to zero as $N \rightarrow +\infty$. We now restrict ourselves to the case $\operatorname{Re} \tau \geq 0$. The argument in the case $\operatorname{Re} \tau < 0$ is completely analogous.

In this case, the integral over the right vertical segment equals

$$i \int_0^h g(N + it) dt = i \int_0^h \frac{\pi e^{i(z-\bar{w})(N+it)} \left(\frac{j+1}{2} - (N + it) \right)}{\sinh((2\beta - \pi)(N + it)) \sinh(\pi(N + it) - \frac{j+1}{2})} dt. \quad (23)$$

Now we write $\tau = u + iv$. Then

$$\operatorname{Re}[i\tau(N + it)] = \operatorname{Re}(i(u + iv)(N + it)) = -ut - vN.$$

Hence the size of the numerator of the integrand in (23) is Ne^{-vN} (as a function of N). As for the denominator, notice that (for a and b real)

$$\sinh(a + ib) = \sinh a \cos b + i \cosh a \sin b,$$

so that

$$|\sinh(a + ib)|^2 = \sinh^2 a + \sin^2 b.$$

Thus we may estimate the size of the denominator in (23) by $C \cdot e^{2\beta N}$.

But $|v| < 2\beta$, so the denominator swamps the numerator as $N \rightarrow \infty$. In conclusion, the integrals over I_N , (and, analogously, the corresponding integral over I_{-N}) disappear for $N \rightarrow +\infty$. The result follows. \square

6. THE SUM OF THE R_j

Recall that, for $\tau \in S_{2\beta}$, the integral defining I_j converges absolutely. Recall also that we decompose I_j as $I_j = R_j + J_j$.

We now define \mathcal{D} to be the domain in \mathbb{C}^2 given by

$$\mathcal{D} = \{(\tau, \lambda) \in \mathbb{C}^2 : |\operatorname{Im} \tau - \log |\lambda|^2| < \pi, e^{-(\beta-\pi/2)} < |\lambda| < e^{(\beta-\pi/2)}\}. \quad (24)$$

Then we have

Proposition 6.1. *Let $c_0 > 0$ be fixed. There exists a function $E(\tau, \lambda) \in C^\infty$ in a neighborhood of $\overline{\mathcal{D}} \cap \{(\tau, \lambda) : |\operatorname{Re} \tau| \geq c_0\}$ in \mathbb{C}^2 such that $D^\alpha E = \mathcal{O}(|\operatorname{Re} \tau|^\alpha)$ as $|\operatorname{Re} \tau| \rightarrow +\infty$ and such that, for $(\tau, \lambda) \in \mathcal{D}$ with $|\operatorname{Re} \tau| \geq c_0$,*

$$\sum_{j \in \mathbb{Z}} R_j(\tau) \lambda^j = e^{-\nu_\beta \tau \operatorname{sgn}(\operatorname{Re} \tau)} \cdot \left(\frac{\varphi_1(\tau)}{(1 - e^{[i\tau - \pi]/2} \lambda)^2} + \frac{\varphi_2(\tau, \lambda)}{(\lambda - e^{-[\pi + i\tau]/2})^2} + E(\tau, \lambda) \right), \quad (25)$$

where

$$\begin{aligned} \varphi_1(\tau) &= \frac{2\nu_\beta^2}{\pi} e^{-i\pi\nu_\beta} e^{[i\tau - \pi]/2} \left(\frac{i}{2} + \nu_\beta (1 - e^{[i\tau - \pi]/2}) \right), \\ \varphi_2(\tau, \lambda) &= \frac{2\nu_\beta^2}{\pi} e^{i\pi\nu_\beta} \lambda e^{-[i\tau + \pi]/2} \left(\nu_\beta e^{-[i\tau + \pi]/2} - \left(\nu_\beta - \frac{i}{2} \right) \lambda \right). \end{aligned}$$

The convergence of the series is uniform on compact subsets of \mathcal{D} .

Proof. We obtain the formula (25) by approximating the sinh function by an exponential, summing the resulting geometric series, and then showing that the performed approximation gives rise to error terms that are more regular than the explicit term obtained by replacing the sinh functions by exponentials.

Notice that, for $a, b \in \mathbb{R}$, $a \neq 0$, $|b| < \pi$,

$$\frac{e^{|a|}}{\sinh(a + ib)} = 2 \operatorname{sgn}(a) e^{-i \operatorname{sgn}(a)b} \left(1 + \frac{e^{-2 \operatorname{sgn}(a)(a+ib)}}{1 - e^{-2 \operatorname{sgn}(a)(a+ib)}} \right). \quad (26)$$

For simplicity of notation we momentarily restrict ourselves to the case $\operatorname{Re} \tau \geq 0$. The argument in the case $\operatorname{Re} \tau < 0$ is analogous. Using Lemma 5.4 and Proposition 5.5, we write

$$\begin{aligned} R_j(\tau) &= \frac{\nu_\beta^2 e^{-\nu_\beta \tau} \left(\nu_\beta + i \frac{j+1}{2} \right) e^{i \frac{j+1}{2} \tau}}{\pi \sinh(\pi \frac{j+1}{2} + i \nu_\beta \pi)} \\ &= \frac{\nu_\beta^2}{\pi} e^{-\nu_\beta \tau} \left[\nu_\beta \frac{e^{i \frac{j+1}{2} \tau}}{\sinh(\pi \frac{j+1}{2} + i \nu_\beta \pi)} + i \frac{j+1}{2} \frac{e^{i \frac{j+1}{2} \tau}}{\sinh(\pi \frac{j+1}{2} + i \nu_\beta \pi)} \right]. \end{aligned}$$

We now use (26). The first sum that we must consider is

$$\begin{aligned}
& \sum_{j \in \mathbb{Z}} \frac{e^{i \frac{j+1}{2} \tau}}{\sinh(\pi \frac{j+1}{2} + i \nu_\beta \pi)} \lambda^j \\
&= 2 \left(\sum_{j \in \mathbb{Z}, j \neq -1} \sigma(j) e^{-i \sigma(j) \pi \nu_\beta} e^{i \frac{j+1}{2} \tau - \pi |\frac{j+1}{2}|} \lambda^j \right. \\
&\quad \left. + \sum_{j \in \mathbb{Z}} \sigma(j) e^{-i \sigma(j) \pi \nu_\beta} e^{i \frac{j+1}{2} \tau} \frac{e^{-3\pi |\frac{j+1}{2}| - 2i \pi \nu_\beta \sigma(j)}}{1 - e^{-2\pi |\frac{j+1}{2}| - 2i \pi \nu_\beta \sigma(j)}} \lambda^j + \frac{1}{\sinh(i \pi \nu_\beta)} \right) \\
&\equiv 2(F_1 + E_1); \tag{27}
\end{aligned}$$

here $\sigma(j) = \text{sgn}(j+1)$. On the other hand, the second sum is

$$\begin{aligned}
& \sum_{j \in \mathbb{Z}} \frac{j+1}{2} \frac{e^{i \frac{j+1}{2} \tau}}{\sinh(\pi \frac{j+1}{2} + i \pi \nu_\beta)} \lambda^j \\
&= 2 \left(\sum_{j \in \mathbb{Z}} \frac{j+1}{2} \sigma(j) e^{-i \sigma(j) \pi \nu_\beta} e^{i \frac{j+1}{2} \tau - \pi |\frac{j+1}{2}|} \lambda^j \right. \\
&\quad \left. + \sum_{j \in \mathbb{Z}} \frac{j+1}{2} \sigma(j) e^{-i \sigma(j) \pi \nu_\beta} e^{i \frac{j+1}{2} \tau} \frac{e^{-3\pi |\frac{j+1}{2}| - 2i \pi \nu_\beta \sigma(j)}}{1 - e^{-\pi |\frac{j+1}{2}| - 2i \pi \nu_\beta \sigma(j)}} \lambda^j \right) \\
&\equiv 2(F_2 + E_2). \tag{28}
\end{aligned}$$

Notice that then

$$\sum_{j \in \mathbb{Z}} R_j(\tau) \lambda^j = \frac{2\nu_\beta^2}{\pi} e^{-\nu_\beta \tau} [\nu_\beta (F_1 + E_1) + i(F_2 + E_2)]. \tag{29}$$

Now

$$\begin{aligned}
F_1 &= \sum_{j \in \mathbb{Z}, j \neq -1} \sigma(j) e^{-i \sigma(j) \pi \nu_\beta} e^{i(\frac{j+1}{2})\tau - \pi |\frac{j+1}{2}|} \lambda^j \\
&= e^{-i \pi \nu_\beta} \sum_{j \geq 0} e^{(i\tau - \pi)(\frac{j+1}{2})} \lambda^j e^{i \pi \nu_\beta} \sum_{j < -1} e^{(i\tau + \pi)\frac{j+1}{2}} \lambda^j \\
&= e^{-i \pi \nu_\beta + [i\tau - \pi]/2} \sum_{j \geq 0} (e^{[i\tau - \pi]/2} \lambda)^j - e^{-i \pi \nu_\beta + [i\tau + \pi]/2} \sum_{j \geq 1} (e^{-[i\tau + \pi]/2} \lambda^{-1})^j + \frac{e^{i \pi \nu_\beta}}{\lambda} \\
&= \frac{e^{-i \pi \nu_\beta + [i\tau - \pi]/2}}{1 - e^{[i\tau - \pi]/2} \lambda} - \frac{e^{i \pi \nu_\beta - [i\tau + \pi]/2}}{\lambda - e^{-[i\tau + \pi]/2}} + \frac{e^{i \pi \nu_\beta}}{\lambda}.
\end{aligned}$$

Here, of course, the first sum converges for $|e^{[i\tau - \pi]/2} \lambda| < 1$ and the second sum converges for $|e^{-[i\tau + \pi]/2} \lambda^{-1}| < 1$. So the full series, summed over $j \in \mathbb{Z}$, converges on the annulus $e^{[\text{Im } \tau - \pi]/2} < |\lambda| < e^{[\text{Im } \tau + \pi]/2}$.

For F_2 , notice that the sum is precisely the same as the one we just computed except for a factor of $\frac{j+1}{2}$ in front. Thus we can formally differentiate in τ and obtain

$$\begin{aligned} F_2 &= \sum_{j \in \mathbb{Z}} \frac{j+1}{2} e^{i(\frac{j+1}{2})\tau - \pi|\frac{j+1}{2}|} \lambda^j = \frac{1}{i} \frac{d}{d\tau} F_1 \\ &= \frac{1}{2} \frac{e^{-i\pi\nu_\beta + [i\tau - \pi]/2}}{(1 - e^{[i\tau - \pi]/2} \lambda)^2} + \frac{1}{2} \frac{\lambda e^{i\pi\nu_\beta - [i\tau + \pi]/2}}{(\lambda - e^{-[i\tau + \pi]/2})^2}. \end{aligned}$$

Therefore

$$\begin{aligned} &\nu_\beta F_1 + iF_2 \\ &= \nu_\beta \left(\frac{e^{-i\pi\nu_\beta + [i\tau - \pi]/2}}{1 - e^{[i\tau - \pi]/2} \lambda} - \frac{e^{i\pi\nu_\beta - [i\tau + \pi]/2}}{\lambda - e^{-[i\tau + \pi]/2}} \right) + \frac{i}{2} \left(\frac{e^{-i\pi\nu_\beta + [i\tau - \pi]/2}}{(1 - e^{[i\tau - \pi]/2} \lambda)^2} \frac{\lambda e^{i\pi\nu_\beta - [i\tau + \pi]/2}}{(\lambda - e^{-[i\tau + \pi]/2})^2} \right) \\ &= \frac{e^{-i\pi\nu_\beta + [i\tau - \pi]/2}}{(1 - e^{[i\tau - \pi]/2} \lambda)^2} \left(\frac{i}{2} + \nu_\beta (1 - e^{[i\tau - \pi]/2}) \right) \\ &\quad + \frac{e^{i\pi\nu_\beta - [i\tau + \pi]/2}}{(\lambda - e^{-[i\tau + \pi]/2})^2} \left(\nu_\beta e^{-[i\tau + \pi]/2} - \left(\nu_\beta - \frac{i}{2} \right) \lambda \right). \end{aligned} \quad (30)$$

Hence, by (29) and (30), and recalling that $\operatorname{Re} \tau \geq c_0 > 0$, we have

$$\begin{aligned} &\sum_{j \in \mathbb{Z}} R_j(\tau) \lambda^j = \\ &e^{-\nu_\beta \tau} \left[\frac{\varphi_1(\tau, \lambda)}{(1 - e^{[i\tau - \pi]/2} \lambda)^2} + \frac{\varphi_2(\tau, \lambda)}{(\lambda - e^{-[i\tau + \pi]/2})^2} \right] + \frac{\nu_\beta^2}{2\pi} e^{-\nu_\beta \tau + i\pi\nu_\beta} [\nu_\beta E_1 + iE_2], \end{aligned} \quad (31)$$

where

$$\begin{aligned} \varphi_1(\tau, \lambda) &= \frac{2\nu_\beta^2}{\pi} e^{-i\pi\nu_\beta} e^{[i\tau - \pi]/2} \left(\frac{i}{2} + \nu_\beta (1 - e^{[i\tau - \pi]/2}) \right), \\ \varphi_2(\tau, \lambda) &= \frac{2\nu_\beta^2}{\pi} e^{i\pi\nu_\beta} \lambda e^{-[i\tau + \pi]/2} \left(\nu_\beta e^{-[i\tau + \pi]/2} - \left(\nu_\beta - \frac{i}{2} \right) \lambda \right), \end{aligned}$$

as we claimed. Notice in particular φ_1, φ_2 are entire functions, and are bounded as $|\operatorname{Re} \tau| \rightarrow +\infty$, together with all their derivatives.

We now study the error term

$$E \equiv \frac{2\nu_\beta^2}{\pi} e^{-\nu_\beta \tau - i\pi\nu_\beta} [\nu_\beta E_1 + iE_2].$$

Notice that there exists a positive constant $c_0 > 0$ such that, for all j ,

$$|1 - e^{-2\pi|\frac{j+1}{2}| - 2i\pi\nu_\beta}| \geq c_0.$$

Hence the series

$$\sum_{j \in \mathbb{Z}} e^{i \frac{j+1}{2} \tau} \frac{e^{-3\pi |\frac{j+1}{2}| - 2i\pi\nu_\beta \sigma(j)}}{1 - e^{-2\pi |\frac{j+1}{2}| - 2i\pi\nu_\beta \sigma(j)}} \lambda^j \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \frac{j+1}{2} e^{i \frac{j+1}{2} \tau} \frac{e^{-3\pi |\frac{j+1}{2}| - 2i\pi\nu_\beta \sigma(j)}}{1 - e^{-2\pi |\frac{j+1}{2}| - 2i\pi\nu_\beta \sigma(j)}} \lambda^j$$

converge when

$$|e^{-[i\tau+3\pi]/2}| < |\lambda| < |e^{-[i\tau-3\pi]/2}|,$$

which is a strictly larger annulus than $e^{[\operatorname{Im} \tau - \pi]/2} < |\lambda| < e^{[\operatorname{Im} \tau + \pi]/2}$. Thus the sums of the two series are smooth and bounded, with all derivatives smooth and bounded, on a neighborhood of the closure $\overline{\mathcal{D}}$ of \mathcal{D} (where this domain is defined at the beginning of Section 6).

For the case $\operatorname{Re} \tau < -c_0$, notice that

$$\begin{aligned} R_j(\tau) &= \frac{\nu_\beta^2 e^{\nu_\beta \tau} (\nu_\beta - i \frac{j+1}{2}) e^{i \frac{j+1}{2} \tau}}{\pi \sinh(\pi \frac{j+1}{2} - i \nu_\beta \pi)} \\ &= \frac{\nu_\beta^2}{\pi} e^{\nu_\beta \tau} \left[\nu_\beta \frac{e^{i \frac{j+1}{2} \tau}}{\sinh(\pi \frac{j+1}{2} - i \nu_\beta \pi)} - i \frac{j+1}{2} \frac{e^{i \frac{j+1}{2} \tau}}{\sinh(\pi \frac{j+1}{2} - i \nu_\beta \pi)} \right]. \end{aligned}$$

Then we consider the sums

$$\begin{aligned} &2 \left(\sum_{j \in \mathbb{Z}} \sigma(j) e^{-i\sigma(j)\pi\nu_\beta} e^{i \frac{j+1}{2} \tau - \pi |\frac{j+1}{2}|} \lambda^j + \sum_{j \in \mathbb{Z}} \sigma(j) e^{-i\sigma(j)\pi\nu_\beta} e^{i \frac{j+1}{2} \tau} \frac{e^{-3\pi |\frac{j+1}{2}| - 2i\sigma(j)\pi\nu_\beta}}{1 - e^{-2\pi |\frac{j+1}{2}| - 2i\sigma(j)\pi\nu_\beta}} \lambda^j \right), \\ &2 \left(\sum_{j \in \mathbb{Z}} \frac{j+1}{2} \sigma(j) e^{-i\sigma(j)\pi\nu_\beta} e^{i \frac{j+1}{2} \tau - \pi |\frac{j+1}{2}|} \lambda^j + \sum_{j \in \mathbb{Z}} \frac{j+1}{2} \sigma(j) e^{-i\sigma(j)\pi\nu_\beta} e^{i \frac{j+1}{2} \tau} \frac{e^{-3\pi |\frac{j+1}{2}| - 2i\sigma(j)\pi\nu_\beta}}{1 - e^{-2\pi |\frac{j+1}{2}| - 2i\sigma(j)\pi\nu_\beta}} \lambda^j \right). \end{aligned}$$

Therefore, when $\operatorname{Re} \tau < c_0$,

$$\sum_{j \in \mathbb{Z}} R_j(\tau) \lambda^j = \frac{2\nu_\beta^2}{\pi} e^{\nu_\beta \tau} [\nu_\beta (F_1 + E'_1) - i(F_2 + E'_2)],$$

where F_1, F_2 are as in the previous case, while E'_1, E'_2 are error terms that are dealt with as in the case $\operatorname{Re} \tau \geq c_0$. \square

7. THE SUM OF THE J_j

We wish to study the sum

$$\sum_{j \in \mathbb{Z}} J_j(\tau) \lambda^j,$$

where

$$J_j(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\tau(\xi+ih)} (\xi+ih) (\xi+ih - \frac{j+1}{2})}{\sinh(\pi(\xi+ih)) \sinh((2\beta-\pi)(\xi+ih - \frac{j+1}{2}))} d\xi.$$

Remark 7.1. We make an important observation. We arrived at the definition of $J_j(\tau)$ by computing a contour integral along the rectangular path γ_N^\pm , whose upper (lower) side is the segment $[-N \pm ih, N \pm ih]$, for $\operatorname{Re} \tau > 0$ ($\operatorname{Re} \tau < 0$) respectively. However, the definition of $J_j(\tau)$ makes sense for all real values of the parameter h , in particular for $h = 0$. Notice that, in this case, $J_j(\tau) = I_j(\tau)$. We shall make use of this fact when we compute the expression of the Bergman kernel for $|\operatorname{Re}(z_1 - \bar{w}_1)| \leq c_0$.

We use (26) to write

$$\begin{aligned}
& \frac{1}{\sinh(\pi(\xi + ih)) \sinh((2\beta - \pi)(\xi + ih - \frac{j+1}{2}))} \\
&= \frac{2\operatorname{sgn}(\xi)e^{-i\operatorname{sgn}(\xi)\pi h}}{e^{\pi|\xi|}} \left(1 + \frac{e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}}{1 - e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}} \right) \\
&\quad \times \frac{2\operatorname{sgn}(\xi - \frac{j+1}{2})e^{-i\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)h}}{e^{(2\beta-\pi)|\xi - \frac{j+1}{2}|}} \left(1 + \frac{e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}}{1 - e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}} \right) \\
&= \frac{4\operatorname{sgn}(\xi)\operatorname{sgn}(\xi - \frac{j+1}{2})e^{-ih(\operatorname{sgn}(\xi)\pi + \operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi))}}{e^{\pi|\xi|}e^{(2\beta-\pi)|\xi - \frac{j+1}{2}|}} \\
&\quad \times \left(1 + \frac{e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}}{1 - e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}} + \frac{e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}}{1 - e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}} \right. \\
&\quad \left. + \frac{e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}}{1 - e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}} \cdot \frac{e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}}{1 - e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}} \right).
\end{aligned}$$

We set

$$\sigma(\xi) = \operatorname{sgn}(\xi)\operatorname{sgn}(\xi - \frac{j+1}{2})e^{-ih(\operatorname{sgn}(\xi)\pi + \operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi))} \quad (32)$$

and decompose J_j accordingly as

$$J_j(\tau) = M_j(\tau) + E_j^{(1)}(\tau) + E_j^{(2)}(\tau) + E_j^{(3)}(\tau), \quad (33)$$

where

$$\begin{aligned}
M_j(\tau) &= \frac{2}{\pi} \int_{-\infty}^{+\infty} \sigma(\xi) e^{i\tau(\xi+ih)} \frac{(\xi + ih)(\xi + ih - \frac{j+1}{2})}{e^{\pi|\xi|}e^{(2\beta-\pi)|\xi - \frac{j+1}{2}|}} d\xi \\
&= \frac{2}{\pi} e^{-h\tau} \int_{-\infty}^{+\infty} \sigma(\xi) e^{i\tau\xi} \frac{(\xi + ih)(\xi + ih - \frac{j+1}{2})}{e^{\pi|\xi|}e^{(2\beta-\pi)|\xi - \frac{j+1}{2}|}} d\xi, \quad (34)
\end{aligned}$$

and, analogously,

$$\begin{aligned}
E_j^{(1)}(\tau) &= \frac{2}{\pi} e^{-h\tau} \int_{-\infty}^{+\infty} \sigma(\xi) e^{i\tau\xi} \frac{(\xi + ih)(\xi + ih - \frac{j+1}{2})}{e^{\pi|\xi|} e^{(2\beta-\pi)|\xi - \frac{j+1}{2}|}} \cdot \frac{e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}}{1 - e^{-2\operatorname{sgn}(\xi)(\xi+ih)}} d\xi, \\
E_j^{(2)}(\tau) &= \frac{2}{\pi} e^{-h\tau} \int_{-\infty}^{+\infty} \sigma(\xi) e^{i\tau\xi} \frac{(\xi + ih)(\xi + ih - \frac{j+1}{2})}{e^{\pi|\xi|} e^{(2\beta-\pi)|\xi - \frac{j+1}{2}|}} \cdot \\
&\quad \times \frac{e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}}{1 - e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}} d\xi \\
E_j^{(3)}(\tau) &= \frac{2}{\pi} e^{-h\tau} \int_{-\infty}^{+\infty} \sigma(\xi) e^{i\tau\xi} \frac{(\xi + ih)(\xi + ih - \frac{j+1}{2})}{e^{\pi|\xi|} e^{(2\beta-\pi)|\xi - \frac{j+1}{2}|}} \cdot \\
&\quad \times \frac{e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}}{1 - e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}} \cdot \frac{e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}}{1 - e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}} d\xi.
\end{aligned} \tag{35}$$

Thus we have reduced the situation to computing the sums

$$\sum_{j \in \mathbb{Z}} M_j(\tau) \lambda^j + \sum_{j \in \mathbb{Z}} E_j^{(1)}(\tau) \lambda^j + \sum_{j \in \mathbb{Z}} E_j^{(2)}(\tau) \lambda^j + \sum_{j \in \mathbb{Z}} E_j^{(3)}(\tau) \lambda^j.$$

We shall prove the following results.

Proposition 7.2. *There exist entire functions ψ_j^\pm , $j = 1, \dots, 4$, which are uniformly (together with all their derivatives) of size $\mathcal{O}(|\operatorname{Re} \tau|)$ as $|\operatorname{Re} \tau| \rightarrow +\infty$ in any set $\{|\operatorname{Im} \tau| \leq C, |\lambda| \leq C'\}$, such that*

$$\begin{aligned}
\sum_{j \in \mathbb{Z}} M_j(\tau) \lambda^j &= \frac{1}{\pi} e^{-\operatorname{sgn}(\operatorname{Re} \tau)h\tau} \left\{ \frac{e^{i2\beta h}}{(i\tau + 2\beta)^2} \left[\frac{e^{(\beta-\pi/2)}}{2(e^{(\beta-\pi/2)} - \lambda)^2} (1 + \psi_1^+(\tau, \lambda)) \right. \right. \\
&\quad \left. \left. + \frac{e^{-[i\tau+\pi]/2}}{2(\lambda - e^{-[i\tau+\pi]/2})^2} (1 + \psi_2^-(\tau, \lambda)) + \frac{e^{-i2\beta h - [i\tau+\pi]/2} \psi_4^+(\tau)}{2(e^{(\beta-\pi/2)} - \lambda)(e^{-[i\tau+\pi]/2} - \lambda)} \right] \right. \\
&\quad \left. + \frac{e^{-i2\beta h}}{(i\tau - 2\beta)^2} \left[\frac{e^{[\pi-i\tau]/2}}{2(e^{[\pi-i\tau]/2} - \lambda)^2} (1 + \psi_2^+(\tau, \lambda)) \right. \right. \\
&\quad \left. \left. + \frac{e^{-(\beta-\pi/2)}}{2(\lambda - e^{-(\beta-\pi/2)})^2} (1 + \psi_1^-(\tau, \lambda)) + \frac{e^{-i2\beta h + [\pi-i\tau]/2} \psi_4^-(\tau)}{2(e^{-(\beta-\pi/2)} - \lambda)(e^{[\pi-i\tau]/2} - \lambda)} \right] \right. \\
&\quad \left. + \frac{e^{ih(2\beta-2\pi) + (\beta-\pi/2)}}{2(e^{[\pi-i\tau]/2} - \lambda)^2 (e^{(\beta-\pi/2)} - \lambda)^2} \psi_3^+(\tau, \lambda) \right. \\
&\quad \left. + \frac{e^{-ih(2\beta-2\pi) - (\beta-\pi/2)}}{2(e^{-[i\tau+\pi]/2} - \lambda)^2 (e^{-(\beta-\pi/2)} - \lambda)^2} \psi_3^-(\tau, \lambda) \right\}.
\end{aligned}$$

Here h is as in (4).

The convergence is uniform on compact subsets of D'_β . Moreover, the functions ψ_j^\pm can be computed explicitly (see (72), (81), (78), (84), and (68) respectively).

Proposition 7.3. *There exist functions $\Psi_1^{(k)}$, $\Psi_2^{(k)}$ and $\Psi_3^{(k)}$, holomorphic in a neighborhood of $\overline{\mathcal{D}}$, bounded together with all their derivatives as $|\operatorname{Re} \tau| \rightarrow +\infty$, such that, for $k = 1, 2, 3$,*

$$\sum_{j \in \mathbb{Z}} E_j^{(k)}(\tau) \lambda^j = e^{-\operatorname{sgn}(\operatorname{Re} \tau) h \tau} \left\{ \frac{\Psi_1^{(k)}(\tau, \lambda)}{(\lambda - e^{-(\beta - \pi/2)})^2} + \frac{\Psi_2^{(k)}(\tau, \lambda)}{(e^{(\beta - \pi/2)} - \lambda)^2} + \Psi_3^{(k)}(\tau, \lambda) \right\}.$$

Here h is as in (4).

(Recall that the $E_j^{(k)}$ are defined in (35).)

We shall compute these sums, and prove these two propositions, in Sections 9 and 10.

8. THE BERGMAN KERNELS FOR D_β AND D'_β – PROOF OF THEOREMS 1 AND 2

In this section, assuming the validity of Propositions 7.2 and 7.3, we complete the proof of the asymptotic expansions for the Bergman kernels for the worm domains D_β and D'_β .

Proof of Theorem 1. Before we move ahead, let us review what we have accomplished, and how all the parts fit together.

From Remark 5.3 we know that the Bergman kernel $K_{D'_\beta}$ for D'_β can be written as

$$K_{D'_\beta}(z_1, z_2, w_1, w_2) = \sum_{j \in \mathbb{Z}} I_j(\tau) \lambda^j,$$

where we set $z_1 - \overline{w}_1 = \tau$ and $z_2 \overline{w}_2 = \lambda$.

When $|\operatorname{Re} \tau| > c_0$ we use the decomposition

$$\sum_j I_j(\tau) \lambda^j = \sum_j R_j(\tau) \lambda^j + \sum_j M_j(\tau) \lambda^j + \sum_{k=1}^3 \sum_j E_j^{(k)}(\tau) \lambda^j.$$

We put together (21), Proposition 5.5, Proposition 6.1, (33), Proposition 7.2, and Proposition 7.3. For $|\operatorname{Re}(z_1 - \bar{w}_1)| > c_0$, we have that

$$\begin{aligned}
& K_{D'_\beta}(z_1, z_2, w_1, w_2) \\
&= e^{-\nu_\beta(z_1 - \bar{w}_1) \operatorname{sgn}(\operatorname{Re}(z_1 - \bar{w}_1))} \left\{ \frac{\varphi_1(z_1 - \bar{w}_1)}{(1 - e^{[i(z_1 - \bar{w}_1) - \pi]/2} z_2 \bar{w}_2)^2} + \frac{\varphi_2(z_1 - \bar{w}_1, z_2 \bar{w}_2)}{(z_2 \bar{w}_2 - e^{-[\pi + i(z_1 - \bar{w}_1)]/2})^2} \right\} \\
&+ \frac{1}{\pi} e^{-\operatorname{sgn}(\operatorname{Re}(z_1 - \bar{w}_1)) h(z_1 - \bar{w}_1)} \left\{ \frac{e^{i2\beta h}}{(i(z_1 - \bar{w}_1) + 2\beta)^2} \left[\frac{e^{(\beta - \pi/2)}}{2(e^{(\beta - \pi/2)} - z_2 \bar{w}_2)^2} (1 + \psi_1^+(z_1 - \bar{w}_1, z_2 \bar{w}_2)) \right. \right. \\
&\quad \left. \left. + \frac{e^{-[i(z_1 - \bar{w}_1) + \pi]/2}}{2(z_2 \bar{w}_2 - e^{-[i(z_1 - \bar{w}_1) + \pi]/2})^2} (1 + \psi_2^-(z_1 - \bar{w}_1, z_2 \bar{w}_2)) \right. \right. \\
&\quad \left. \left. + \frac{e^{-i2\beta h - [i(z_1 - \bar{w}_1) + \pi]/2} \psi_4^+(z_1 - \bar{w}_1)}{2(e^{(\beta - \pi/2)} - z_2 \bar{w}_2)(e^{-[i(z_1 - \bar{w}_1) + \pi]/2} - z_2 \bar{w}_2)} \right] \right. \\
&\quad \left. + \frac{e^{-i2\beta h}}{(i(z_1 - \bar{w}_1) - 2\beta)^2} \left[\frac{e^{[\pi - i(z_1 - \bar{w}_1)]/2}}{2(e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2)^2} (1 + \psi_2^+(z_1 - \bar{w}_1, z_2 \bar{w}_2)) \right. \right. \\
&\quad \left. \left. + \frac{e^{-(\beta - \pi/2)}}{2(z_2 \bar{w}_2 - e^{-(\beta - \pi/2)})^2} (1 + \psi_1^-(z_1 - \bar{w}_1, z_2 \bar{w}_2)) \right. \right. \\
&\quad \left. \left. + \frac{e^{-i2\beta h + [\pi - i(z_1 - \bar{w}_1)]/2} \psi_4^-(z_1 - \bar{w}_1)}{2(e^{-(\beta - \pi/2)} - z_2 \bar{w}_2)(e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2)} \right] \right. \\
&\quad \left. + \frac{e^{ih(2\beta - 2\pi) + (\beta - \pi/2)}}{2(e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2)^2 (e^{(\beta - \pi/2)} - z_2 \bar{w}_2)^2} \psi_3^+(z_1 - \bar{w}_1, z_2 \bar{w}_2) \right. \\
&\quad \left. + \frac{e^{-ih(2\beta - 2\pi) - (\beta - \pi/2)}}{2(e^{-[i(z_1 - \bar{w}_1) + \pi]/2} - z_2 \bar{w}_2)^2 (e^{-(\beta - \pi/2)} - z_2 \bar{w}_2)^2} \psi_3^-(z_1 - \bar{w}_1, z_2 \bar{w}_2) \right. \\
&\quad \left. + \frac{\Psi_1(z_1 - \bar{w}_1, z_2 \bar{w}_2)}{(z_2 \bar{w}_2 - e^{-(\beta - \pi/2)})^2} + \frac{\Psi_2(z_1 - \bar{w}_1, z_2 \bar{w}_2)}{(e^{(\beta - \pi/2)} - z_2 \bar{w}_2)^2} + \Psi_3(z_1 - \bar{w}_1, z_2 \bar{w}_2) \right\}, \quad (36)
\end{aligned}$$

where $\Psi_j \equiv \sum_{k=1}^3 \Psi_j^{(k)}$. We recall that h is a fixed parameter, $\nu_\beta < h < \min(1, 2\nu_\beta)$, and that the functions ψ_j^\pm s, φ s and Ψ s depend (smoothly) on h . Moreover, we recall that these functions all satisfy the condition $\mathcal{O}(|\operatorname{Re} z_1 - \operatorname{Re} w_1|)$, together with all their derivatives, for $z, w \in \overline{D'_\beta}$.

For $|\operatorname{Re}(z_1 - w_1)| \leq c_0$ we use the result from Section 7, with $h = 0$, for which see Remark 7.1. By Propositions 7.2 and 7.3, with $h = 0$, we have that when

$$|\operatorname{Re}(z_1 - w_1)| \leq c_0,$$

$$\begin{aligned}
& K_{D'_\beta}(z_1, z_2, w_1, w_2) \\
&= \frac{e^{(\beta-\pi/2)}(1 + \psi_1^+(z_1 - \bar{w}_1, z_2 \bar{w}_2))}{2(i(z_1 - \bar{w}_1) + 2\beta)^2(e^{(\beta-\pi/2)} - z_2 \bar{w}_2)^2} \\
&\quad + \frac{e^{-[i(z_1 - \bar{w}_1) + \pi]/2}(1 + \psi_2^-(z_1 - \bar{w}_1, z_2 \bar{w}_2))}{2(i(z_1 - \bar{w}_1) + 2\beta)^2(z_2 \bar{w}_2 - e^{-[i(z_1 - \bar{w}_1) + \pi]/2})} \\
&\quad + \frac{e^{-[i(z_1 - \bar{w}_1) + \pi]/2}\psi_4^+(z_1 - \bar{w}_1)}{4(i(z_1 - \bar{w}_1) + 2\beta)^2(e^{(\beta-\pi/2)} - z_2 \bar{w}_2)(e^{-[i(z_1 - \bar{w}_1) + \pi]/2} - z_2 \bar{w}_2)} \\
&\quad + \frac{e^{[\pi - i(z_1 - \bar{w}_1)]/2}(1 + \psi_2^+(z_1 - \bar{w}_1, z_2 \bar{w}_2))}{2(i(z_1 - \bar{w}_1) - 2\beta)^2(e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2)^2} \\
&\quad + \frac{e^{-(\beta-\pi/2)}(1 + \psi_1^-(z_1 - \bar{w}_1, z_2 \bar{w}_2))}{2(i(z_1 - \bar{w}_1) - 2\beta)^2(z_2 \bar{w}_2 - e^{-(\beta-\pi/2)})^2} \\
&\quad + \frac{e^{[\pi - i(z_1 - \bar{w}_1)]/2}\psi_4^-(z_1 - \bar{w}_1)}{4(i(z_1 - \bar{w}_1) - 2\beta)^2(e^{-(\beta-\pi/2)} - z_2 \bar{w}_2)(e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2)} \\
&\quad + \frac{e^{(\beta-\pi/2)}\psi_3^+(z_1 - \bar{w}_1, z_2 \bar{w}_2)}{2(e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2)^2(e^{(\beta-\pi/2)} - z_2 \bar{w}_2)^2} \\
&\quad + \frac{e^{-(\beta-\pi/2)}\psi_3^-(z_1 - \bar{w}_1, z_2 \bar{w}_2)}{2(e^{-[i(z_1 - \bar{w}_1) + \pi]/2} - z_2 \bar{w}_2)^2(e^{-(\beta-\pi/2)} - z_2 \bar{w}_2)^2} + E(z_1 - \bar{w}_1, z_2 \bar{w}_2). \tag{37}
\end{aligned}$$

$$\begin{aligned}
& + \frac{e^{[\pi - i(z_1 - \bar{w}_1)]/2}\psi_4^-(z_1 - \bar{w}_1)}{4(i(z_1 - \bar{w}_1) - 2\beta)^2(e^{-(\beta-\pi/2)} - z_2 \bar{w}_2)(e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2)} \\
& + \frac{e^{(\beta-\pi/2)}\psi_3^+(z_1 - \bar{w}_1, z_2 \bar{w}_2)}{2(e^{[\pi - i(z_1 - \bar{w}_1)]/2} - z_2 \bar{w}_2)^2(e^{(\beta-\pi/2)} - z_2 \bar{w}_2)^2} \\
& + \frac{e^{-(\beta-\pi/2)}\psi_3^-(z_1 - \bar{w}_1, z_2 \bar{w}_2)}{2(e^{-[i(z_1 - \bar{w}_1) + \pi]/2} - z_2 \bar{w}_2)^2(e^{-(\beta-\pi/2)} - z_2 \bar{w}_2)^2} + E(z_1 - \bar{w}_1, z_2 \bar{w}_2). \tag{38}
\end{aligned}$$

Here E denotes a function which is holomorphic (z_1, z_2) and anti-holomorphic in (w_1, w_2) , for (z_1, z_2) and (w_1, w_2) varying in a neighborhood of D'_β .

We should point out that the functions ψ_j^\pm in the formula above differ from the ones appearing in (36), since they are obtained from Propositions 7.2 and 7.3 for different values of the parameter h .

Now, from (36) and (38), the proof of Theorem 1 follows at once. \square

Proof of Theorem 2. We now derive the explicit expression for the asymptotic expansion of the Bergman kernel for the worm domain D_β . We use the formula from Theorem 1, together with the transformation formula under biholomorphic mappings (see [Kr1]). Specifically, recall that the domains D'_β and D_β are biholomorphic via the mapping

$$\begin{aligned}
\Phi : D'_\beta &\rightarrow D_\beta \\
(z_1, z_2) &\mapsto (e^{z_1}, z_2) \equiv (\zeta_1, \zeta_2).
\end{aligned}$$

Hence

$$\begin{aligned} \Phi^{-1} : \quad D_\beta &\longrightarrow D'_\beta \\ (\omega_1, \omega_2) &\mapsto (\log \omega_1, \omega_2) = (w_1, w_2). \end{aligned}$$

We see that

$$K_{D_\beta}((\zeta_1, \zeta_2), (\omega_1, \omega_2)) = K_{D'_\beta}(\Phi^{-1}(\zeta), \Phi^{-1}(\omega)) \cdot \det \text{Jac } \Phi^{-1}(\zeta) \overline{\det \text{Jac } \Phi^{-1}(\omega)}.$$

Notice that, on D_β , the function $\log \zeta_1$ is well defined once ζ_2 is fixed, since ζ_1 lies in the half-plane $\text{Re}(\zeta_1 e^{i \log |\zeta_2|^2}) > 0$. Hence, for $\zeta = (\zeta_1, \zeta_2) \in D_\beta$, the number $\log \zeta_1$ is the unique value obtained by analytic continuation starting from the principal branch of the log when ζ_2 lies on the positive real line. Also, observe that, via the transformation Φ ,

$$(z_1 - \bar{w}_1) \mapsto \log(\zeta_1/\bar{w}_1) \quad \text{and} \quad e^{[i(z_1 - \bar{w}_1) + \pi]/2} \mapsto \left(\frac{\zeta_1}{\bar{w}_1}\right)^{i/2} e^{\pi/2},$$

where, by $\log(\zeta_1/\bar{w}_1)$ we mean $\log \zeta_1 - \log \bar{w}_1$.

Thus, since $\text{Jac } \Phi^{-1}(\omega) = 1/\omega_1$, writing $t = |\zeta_1| - |\omega_1|$, we have that

$$\begin{aligned} K_{D_\beta}(\zeta, \omega) &= \frac{1}{\zeta_1 \bar{w}_1} K_{D'_\beta}((\log \zeta_1, \zeta_2), (\log \omega_1, \omega_2)) \\ &= \frac{\chi_1(t)}{\zeta_1 \bar{w}_1} K_b((\log \zeta_1, \zeta_2), (\log \omega_1, \omega_2)) \\ &\quad + \frac{\chi_2(t)}{\zeta_1 \bar{w}_1} \left\{ e^{-h|\log(|\zeta_1|/|\bar{w}_1|)|} e^{-h \text{sgnt} \cdot (\arg \zeta_1 + \arg \omega_1)} K_{\tilde{b}}((\log \zeta_1, \zeta_2), (\log \omega_1, \omega_2)) \right. \\ &\quad + e^{-\nu_b |\log(|\zeta_1|/|\bar{w}_1|)|} e^{-\nu_b \text{sgnt} \cdot (\arg \zeta_1 + \arg \omega_1)} \frac{\phi_1(\log \zeta_1, \omega_1)}{((\zeta_1/\bar{w}_1)^{-i/2} e^{\pi/2} - z_2 \bar{w}_2)^2} \\ &\quad \left. + e^{-\nu_b |\log(|\zeta_1|/|\bar{w}_1|)|} e^{-\nu_b \text{sgnt} \cdot (\arg \zeta_1 + \arg \omega_1)} \frac{\phi_2((\log \zeta_1, \zeta_2), (\omega_1, \omega_2))}{((\zeta_1/\bar{w}_1)^{-i/2} e^{-\pi/2} - z_2 \bar{w}_2)^2} \right\} \\ &= \chi_1(t) \frac{K_b((\log \zeta_1, \zeta_2), (\log \omega_1, \omega_2))}{\zeta_1 \bar{w}_1} \\ &\quad + \chi_2(t) \left\{ \left(\frac{|\zeta_1|}{|\omega_1|} \right)^{-h \text{sgnt}} e^{-h \text{sgnt} \cdot (\arg \zeta_1 + \arg \omega_1)} \frac{K_{\tilde{b}}((\log \zeta_1, \zeta_2), (\log \omega_1, \omega_2))}{\zeta_1 \bar{w}_1} \right. \\ &\quad + \left(\frac{|\zeta_1|}{|\omega_1|} \right)^{-\nu_b \text{sgnt}} e^{-\nu_b \text{sgnt} \cdot (\arg \zeta_1 + \arg \omega_1)} \left(\frac{\phi_1(\log \zeta_1, \omega_1)}{\zeta_1 \bar{w}_1} \cdot \frac{1}{((\zeta_1/\bar{w}_1)^{-i/2} e^{\pi/2} - z_2 \bar{w}_2)^2} \right. \\ &\quad \left. \left. + \frac{\phi_2((\log \zeta_1, \zeta_2), (\omega_1, \omega_2))}{\zeta_1 \bar{w}_1} \cdot \frac{1}{((\zeta_1/\bar{w}_1)^{-i/2} e^{-\pi/2} - z_2 \bar{w}_2)^2} \right) \right\}. \quad (39) \end{aligned}$$

This is the promised expression for the Bergman kernel of D_β . We only need to check that the resulting functions $g_1, g_2, G_1, G_2, \dots, \tilde{G}_1, \tilde{G}_2, \dots$ have the announced

properties. Denoting by G any of these functions, then

$$G(\zeta, \omega) = \psi(\log(\zeta_1/\overline{\omega}_1), \zeta_2\overline{\omega}_2) ,$$

where ψ denotes any of the functions ψ s in Proposition 9.1, or Ψ s in Proposition 7.3. Then it is easy to see that $G \in \mathcal{C}^\infty(\overline{D_\beta} \setminus \{0\} \times \overline{D_\beta} \setminus \{0\})$ and that

$$\partial_\zeta^\alpha \partial_{\overline{\omega}}^\gamma G(\zeta, \omega) = \mathcal{O}(|\zeta_1|^{-|\alpha|} |\omega_1|^{-|\gamma|}) \quad \text{as } |\zeta_1|, |\omega_1| \rightarrow 0 . \quad \square$$

9. FOUNDATIONAL STEPS IN THE PROOFS OF PROPOSITIONS 7.2 AND 7.3

The core of our calculation for the expansion of the Bergman kernel is contained in the following result. We begin the proof of this proposition now, but postpone the completion of its proof until the end of the paper—see Section 11.

Proposition 9.1. *Let $R, S > 0$ and let $\mathcal{D}_{R,S}$ be the domain in \mathbb{C}^2 defined by*

$$\mathcal{D}_{R,S} = \{(\tau, \lambda) \in \mathbb{C}^2 : |\operatorname{Im} \tau - \log |\lambda|^2| < S, e^{-R/2} < |\lambda| < e^{R/2}\} .$$

Let

$$\mathcal{I}_j(\tau) = \int_{-\infty}^{\infty} \sigma_{R,S}(\xi) [\xi^2 + b\xi + c] e^{i\tau\xi} e^{-R|\xi - \frac{j+1}{2}|} e^{-S|\xi|} d\xi ,$$

where

$$\begin{aligned} \sigma_{R,S}(\xi) &= \operatorname{sgn}(\xi) \operatorname{sgn}(\xi - \frac{j+1}{2}) e^{-ih(\operatorname{sgn}(\xi - \frac{j+1}{2})R + \operatorname{sgn}(\xi)S)} , \\ b &= 2ih - \frac{j+1}{2}, \quad \text{and} \quad c = ih(ih - \frac{j+1}{2}) . \end{aligned} \quad (40)$$

Then there exist entire functions ψ_j^\pm , $j = 1, \dots, 4$, uniformly $\mathcal{O}(|\operatorname{Re} \tau|)$ as $|\operatorname{Re} \tau| \rightarrow +\infty$ in any set $\{|\operatorname{Im} \tau| \leq C, |\lambda| \leq C'\}$, together with all their derivatives, such that

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \mathcal{I}_j(\tau) \lambda^j &= \frac{e^{ih(R+S)}}{(i\tau + R + S)^2} \left[\frac{e^{R/2}}{2(e^{R/2} - \lambda)^2} (1 + \psi_1^+(\tau, \lambda)) \right. \\ &\quad \left. + \frac{e^{-[i\tau+S]/2}}{2(\lambda - e^{-[i\tau+S]/2})^2} (1 + \psi_2^-(\tau, \lambda)) + \frac{e^{-[i\tau+S]/2} \psi_4^+(\tau)}{2(e^{R/2} - \lambda)(e^{-[i\tau+S]/2} - \lambda)} \right] \\ &\quad + \frac{e^{-ih(R+S)}}{(i\tau - R - S)^2} \left[\frac{e^{[S-i\tau]/2}}{2(e^{[S-i\tau]/2} - \lambda)^2} (1 + \psi_2^+(\tau, \lambda)) \right. \\ &\quad \left. + \frac{e^{-R/2}}{2(\lambda - e^{-R/2})^2} (1 + \psi_1^-(\tau, \lambda)) + \frac{e^{[S-i\tau]/2} \psi_4^-(\tau)}{2(e^{-R/2} - \lambda)(e^{[S-i\tau]/2} - \lambda)} \right] \\ &\quad + \frac{e^{ih(R-S)+R/2}}{2(e^{[S-i\tau]/2} - \lambda)^2 (e^{R/2} - \lambda)^2} \psi_3^+(\tau, \lambda) \\ &\quad + \frac{e^{-ih(R-S)-R/2}}{2(e^{-[i\tau+S]/2} - \lambda)^2 (e^{-R/2} - \lambda)^2} \psi_3^-(\tau, \lambda) . \end{aligned}$$

The convergence is uniform on compact subsets of $\mathcal{D}_{R,S}$. Moreover, the functions ψ_j^\pm can be computed explicitly (see (72), (81), (78), (84), and (68) respectively).

Basic Steps in the Proof of Proposition 9.1. The proof of Proposition 9.1 is broken up into Lemmas 9.2–9.5. We enunciate those lemmas here and complete the first part of the proof. The proofs of those technical lemmas are deferred until Section 11. Also the concluding parts of the proof of Proposition 9.1 will be presented at that time.

We wish to compute the sum $\sum_{j \in \mathbb{Z}} \mathcal{I}_j(\tau) \lambda^j$, for $(\tau, \lambda) \in \mathcal{D}_{R,S}$.

We begin with the terms having index $j \geq 0$. Writing $\sigma_{R,S}$ explicitly, the integral \mathcal{I}_j becomes

$$\begin{aligned}
& \int_{-\infty}^{\infty} \operatorname{sgn}(\xi) \operatorname{sgn}\left(\xi - \frac{j+1}{2}\right) e^{-ih\left(\operatorname{sgn}(\xi - \frac{j+1}{2})R + \operatorname{sgn}(\xi)S\right)} [\xi^2 + b\xi + c] e^{i\tau\xi} e^{-R|\xi - \frac{j+1}{2}|} e^{-S|\xi|} d\xi \\
&= e^{ih(R+S)} e^{-R\frac{j+1}{2}} \left(\int_{-\infty}^{-\delta} [\xi^2 + b\xi + c] e^{(i\tau+R+S)\xi} d\xi \right. \\
&\quad \left. + \int_{-\delta}^{\delta} [\xi^2 + b\xi + c] e^{(i\tau+R+S)\xi} d\xi \right) \\
&\quad - e^{ih(R-S)} e^{-R\frac{j+1}{2}} \int_{\delta}^{\frac{j+1}{2}-\delta} [\xi^2 + b\xi + c] e^{(i\tau+R-S)\xi} d\xi \\
&\quad + e^{-ih(R+S)} e^{R\frac{j+1}{2}} \left(\int_{\frac{j+1}{2}-\delta}^{\frac{j+1}{2}+\delta} [\xi^2 + b\xi + c] e^{(i\tau-R-S)\xi} d\xi \right. \\
&\quad \left. + \int_{\frac{j+1}{2}+\delta}^{+\infty} [\xi^2 + b\xi + c] e^{(i\tau-R-S)\xi} d\xi \right) \\
&\equiv I + \mathcal{E}_1 - II + \mathcal{E}_2 + III . \tag{41}
\end{aligned}$$

Here δ denotes a non-negative parameter. While in the course of the proof of Proposition 9.1 the parameter δ is unnecessary (and it will be taken to be 0), we need this further refinement in the proof of Proposition 7.3 in order to control certain error terms. At this stage we concentrate our attention on the terms I, II and III and we will deal with the remaining terms at a later time.

We shall use the following trivial fact from calculus:

$$\int (\xi^2 + b\xi + c) e^{\alpha\xi} d\xi = \left[\frac{\xi^2}{\alpha} + \left(\frac{b}{\alpha} - \frac{2}{\alpha^2} \right) \xi + \frac{c}{\alpha} - \frac{b}{\alpha^2} + \frac{2}{\alpha^3} \right] e^{\alpha\xi} . \tag{42}$$

Notice that $(\tau, \lambda) \in \mathcal{D}_{R,S}$ implies that $|\operatorname{Im} \tau| < R + S$. Then the evaluation of our integrals at $\pm\infty$ will always be 0.

We break up the proof into a series of lemmas.

Lemma 9.2. *With the notation above, there exist entire functions ψ_1^+ and ψ_2^+ , uniformly $\mathcal{O}(|\operatorname{Re} \tau|)$ as $|\operatorname{Re} \tau| \rightarrow +\infty$ in any fixed compact set, together with all their derivatives, such that*

$$\begin{aligned} \sum_{j \geq 0} (I + III) \lambda^j &= \frac{e^{ih(R+S)+R/2}}{2(e^{R/2} - \lambda)^2 (i\tau + R + S)^2} e^{-\delta[i\tau + R + S]} (1 + \psi_1^+(\tau, \lambda)) \\ &+ \frac{e^{-ih(R+S)+[S-i\tau]/2}}{2(e^{[S-i\tau]/2} - \lambda)^2 (i\tau - R - S)^2} e^{-\delta[R+S-i\tau]} (1 + \psi_2^+(\tau, \lambda)) \\ &+ \left(\frac{2e^{ih(R+S)-\delta[i\tau + R + S]}}{(e^{R/2} - \lambda)(i\tau + R + S)^3} - \frac{2e^{-ih(R+S)-\delta[R+S-i\tau]}}{(e^{[S-i\tau]/2} - \lambda)(i\tau - R - S)^3} \right). \end{aligned}$$

Moreover, for any $M > 0$, there exist constants C_M such that, for all (τ, λ) such that $|\operatorname{Im} \tau|, |\lambda| \leq M$, as $|\operatorname{Re} \tau|, R, S$ tend to $+\infty$, we have

$$|\psi_j^+(\tau, \lambda)| \leq C_M (|\operatorname{Re} \tau| + R + S) \quad \text{for } j = 1, 2.$$

Lemma 9.3. *There exists an entire function ψ_3^+ , uniformly $\mathcal{O}(1)$ as $|\operatorname{Re} \tau| \rightarrow +\infty$ in any fixed compact set, together with all its derivatives, such that*

$$\sum_{j \geq 0} II \lambda^j = \frac{e^{ih(R-S)-\delta[S-R-i\tau]+R/2}}{2(e^{[S-i\tau]/2} - \lambda)^2 (e^{R/2} - \lambda)^2} \psi_3^+(\tau, \lambda).$$

Moreover, $|\psi_3^+(\tau, \lambda)| \leq C_M (e^{(R+S)/2} + e^S)$, for (τ, λ) in any fixed compact set, as $R, S \rightarrow +\infty$.

Now we turn to the case $j \leq 0$. In this situation the integral $\mathcal{I}_j(\tau)$ equals

$$\begin{aligned} &\int_{-\infty}^{\infty} \operatorname{sgn}(\xi) \operatorname{sgn}(\xi - \frac{j+1}{2}) e^{-ih(\operatorname{sgn}(\xi - \frac{j+1}{2})R + \operatorname{sgn}(\xi)S)} [\xi^2 + b\xi + c] e^{i\tau\xi - R|\xi - \frac{j+1}{2}| - S|\xi|} d\xi \\ &= e^{ih(R+S)} e^{-R(\frac{j+1}{2})} \left(\int_{-\infty}^{\frac{j+1}{2}-\delta} [\xi^2 + b\xi + c] e^{(i\tau + R + S)\xi} d\xi \right. \\ &\quad \left. \int_{\frac{j+1}{2}-\delta}^{\frac{j+1}{2}+\delta} [\xi^2 + b\xi + c] e^{(i\tau - R - S)\xi} d\xi \right) \\ &\quad - e^{ih(S-R)} e^{R(\frac{j+1}{2})} \int_{\frac{j+1}{2}+\delta}^{-\delta} [\xi^2 + b\xi + c] e^{(i\tau - R + S)\xi} d\xi \\ &\quad + e^{-ih(R+S)} e^{R(\frac{j+1}{2})} \left(\int_{-\delta}^{\delta} [\xi^2 + b\xi + c] e^{(i\tau - R - S)\xi} d\xi \right. \\ &\quad \left. \int_{\delta}^{+\infty} [\xi^2 + b\xi + c] e^{(i\tau - R - S)\xi} d\xi \right) \\ &\equiv I^* + \mathcal{E}_1^* - II^* + \mathcal{E}_2^* + III^*. \end{aligned} \tag{43}$$

Here, as in the case $j \geq 0$, we fix (the same) non-negative parameter δ . For the proof of Proposition 9.1 we in fact take $\delta = 0$. Notice that this decomposition makes sense

when $j < -1$ and δ is a small positive parameter. The case $j = -1$ is somewhat special, and can be treated along the same lines. In particular, one can simply take $\delta = 0$ in this case as well.

Lemma 9.4. *There exist entire functions ψ_1^- and ψ_2^- , uniformly $\mathcal{O}(|\operatorname{Re} \tau|)$ as $|\operatorname{Re} \tau| \rightarrow +\infty$ in any fixed compact set, together with all their derivatives, such that*

$$\begin{aligned} \sum_{j < 0} (I^* + III^*) \lambda^j &= \frac{e^{-ih(R+S)-R/2}}{2(\lambda - e^{-R/2})^2(i\tau - R - S)^2} e^{-\delta[R+S-i\tau]} (1 + \psi_1^-(\tau, \lambda)) \\ &+ \frac{e^{ih(R+S)} e^{-[i\tau+S]/2}}{2(\lambda - e^{-[i\tau+S]/2})^2(i\tau + R + S)^2} e^{-\delta[i\tau+R+S]} (1 + \psi_2^-(\tau, \lambda)) \\ &+ \left(\frac{2e^{-ih(R+S)-\delta[R+S-i\tau]}}{(e^{-R/2} - \lambda)(i\tau - R - S)^3} - \frac{2e^{ih(R+S)-\delta[i\tau+R+S]}}{(e^{-[i\tau+S]/2} - \lambda)(i\tau + R + S)^3} \right). \end{aligned}$$

Moreover, for any $M > 0$, there exist constants C_M such that, for all (τ, λ) such that $|\operatorname{Im} \tau|, |\lambda| \leq M$, as $|\operatorname{Re} \tau|, R, S$ tend to $+\infty$, we have

$$|\psi_j^-(\tau, \lambda)| \leq C_M(|\operatorname{Re} \tau| + R + S) \quad \text{for } j = 1, 2.$$

Lemma 9.5. *There exists an entire function ψ_3^- , uniformly $\mathcal{O}(1)$ as $|\operatorname{Re} \tau| \rightarrow +\infty$ in any fixed compact set, together with all its derivatives, such that*

$$\sum_{j < 0} II^* \lambda^j = \frac{e^{-ih(R-S)+\delta[i\tau-R+S]-R/2}}{2(e^{-[i\tau+S]/2} - \lambda)^2(e^{-R/2} - \lambda)^2} \psi_3^-(\tau, \lambda).$$

Moreover, for any $M > 0$, there exists $C_M > 0$ such that for (τ, λ) satisfying $|\tau| \leq M$, $|\lambda| \leq e^M$, we have

$$|\psi_3^-(\tau, \lambda)| \leq C_M(e^{R+S/2} + e^{S(2\delta+1/2)} + e^{2R}),$$

as $R, S \rightarrow +\infty$.

These four lemmas constitute the first part of the proof of Proposition 9.1. We postpone the proof of Lemmas 9.2–9.5 to Section 11, and complete the proof of Proposition 9.1 at that time.

10. PROOF OF PROPOSITIONS 7.2 AND 7.3

Assuming Proposition 9.1 for the moment, we are now in a position to compute the sum of the main terms—Proposition 7.2—and of the error terms—Proposition 7.3.

Proof of Proposition 7.2. It suffices to notice that

$$M_j(\tau) = \frac{2}{\pi} e^{-\operatorname{sgn}(\operatorname{Re} \tau) h \tau} \mathcal{I}_j(\tau),$$

where $\mathcal{I}_j(\tau)$ is as in Proposition 9.1 with $R = 2\beta - \pi$ and $S = \pi$. Moreover, the functions ψ_j^\pm , $j = 1, \dots, 4$ can be obtained by (72), (81), (78), (84) and (68) by taking the same values for R and S as above. \square

Proof of Proposition 7.3. We now compute the sum of the error terms, that is we prove Proposition 7.3. Recall that these error terms arose from the decomposition of $J_j(\tau)$ in Section 7.

We wish to evaluate the three sums $\sum_{j \in \mathbb{Z}} E_j^{(k)}(\tau) \lambda^j$, for $k = 1, 2, 3$.

10.1. Sum of the $E_j^{(1)}$. We recall that, when $\operatorname{Re} \tau \geq 0$,

$$E_j^{(1)}(\tau) = \frac{2}{\pi} e^{-h\tau} \int_{-\infty}^{+\infty} \sigma(\xi) e^{i\tau\xi} \frac{(\xi + ih)(\xi + ih - \frac{j+1}{2})}{e^{\pi|\xi|} e^{(2\beta-\pi)|\xi - \frac{j+1}{2}|}} \cdot \frac{e^{-2\pi \operatorname{sgn}(\xi)(\xi+ih)}}{1 - e^{-2\pi \operatorname{sgn}(\xi)(\xi+ih)}} d\xi,$$

where σ is defined in (32). At this time, for simplicity of notation, we concentrate on the case $\operatorname{Re} \tau \geq 0$. The case $\operatorname{Re} \tau < 0$ will follow by a completely analogous argument.

We decompose the integral defining $E_j^{(1)}$ as in (41) and (43), according to whether $j \geq 0$ or $j < 0$. Then, for a fixed $\delta > 0$,

$$E_j^{(1)}(\tau) = \frac{2}{\pi} e^{-h\tau} (I - II + III + \mathcal{E}_1 + \mathcal{E}_2) \quad (44)$$

when $j \geq 0$, and

$$E_j^{(1)}(\tau) = \frac{2}{\pi} e^{-h\tau} (I^* - II^* + III^* + \mathcal{E}_1^* + \mathcal{E}_2^*) \quad (45)$$

when $j < 0$.

10.2. The Cases of $\sum_{j \geq 0} \mathcal{E}_1 \lambda^j$ and $\sum_{j \geq 0} \mathcal{E}_2 \lambda^j$. We begin by considering the sum over positive indices $\sum_{j \geq 0} \mathcal{E}_1 \lambda^j$ (recall that \mathcal{E}_1 depends on j and τ). Writing $m_1(\xi) = e^{-2\pi \operatorname{sgn}(\xi)(\xi+ih)} / (1 - e^{-2\pi \operatorname{sgn}(\xi)(\xi+ih)})$, we have that

$$\begin{aligned} & \sum_{j \geq 0} \mathcal{E}_1 \lambda^j \\ &= \sum_{j \geq 0} \left(\int_{-\delta}^{\delta} \sigma(\xi) (\xi + ih) (\xi + ih - \frac{j+1}{2}) e^{i\tau\xi} e^{-\pi|\xi|} e^{-(2\beta-\pi)|\xi - \frac{j+1}{2}|} m_1(\xi) d\xi \right) \lambda^j \\ &= \sum_{j \geq 0} e^{-(2\beta-\pi)\frac{j+1}{2}} \left(e^{2\beta ih} \int_{-\delta}^0 (\xi + ih)^2 e^{(i\tau+2\beta)\xi} m_1(\xi) d\xi - \int_0^{\delta} (\xi + ih)^2 e^{(i\tau+2\beta-2\pi)\xi} m_1(\xi) d\xi \right. \\ & \quad \left. - \frac{j+1}{2} \int_{-\delta}^0 (\xi + ih) e^{(i\tau+2\beta)\xi} m_1(\xi) d\xi + \frac{j+1}{2} \int_0^{\delta} (\xi + ih) e^{(i\tau+2\beta-2\pi)\xi} m_1(\xi) d\xi \right) \lambda^j. \end{aligned}$$

Of course we must sum in j . We know that, when $|e^{-\gamma/2}\lambda| < 1$,

$$\sum_{j=0}^{\infty} e^{-\gamma \frac{j+1}{2}} \lambda^j = \frac{1}{e^{\gamma/2} - \lambda} \quad \text{and} \quad \sum_{j=0}^{\infty} e^{-\gamma \frac{j+1}{2}} \frac{j+1}{2} \lambda^j = \frac{e^{\gamma/2}}{2(e^{\gamma/2} - \lambda)^2}. \quad (46)$$

Therefore

$$\sum_{j \geq 0} \mathcal{E}_1 \lambda^j = \frac{h_1(\tau)}{e^{(\beta-\pi/2)} - \lambda} + \frac{h_2(\tau)}{(e^{(\beta-\pi/2)} - \lambda)^2} = \frac{h_1(\tau)(e^{(\beta-\pi/2)} - \lambda) + h_2(\tau)}{(e^{(\beta-\pi/2)} - \lambda)^2},$$

where h_1, h_2 are entire functions of *exponential type*, that is, h_1 and h_2 decay exponentially, together with all their derivatives, in every closed horizontal strip.

Now we turn to the sum $\sum_{j \geq 0} \mathcal{E}_2 \lambda^j$. Now we notice that $m_1(\xi) = \sum_{k=1}^{+\infty} e^{-2k\pi(\xi+ih)}$, for $\xi > 0$, where m_1 is as before. Notice that the series converges uniformly on compact sets, with bounds uniform in $j \geq 0$, so we can interchange the order of integration and summation. Then

$$\begin{aligned} \sum_{j \geq 0} \mathcal{E}_2 \lambda^j &= \sum_{j \geq 0} \left(e^{-2\beta ih} \int_{\frac{j+1}{2}-\delta}^{\frac{j+1}{2}+\delta} (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) e^{i\tau\xi - \pi\xi - (2\beta-\pi)(\xi - \frac{j+1}{2})} m_1(\xi) d\xi \right) \lambda^j \\ &= e^{-(2\beta-\pi)ih} \sum_{j \geq 0} \sum_{k \geq 1} \left(\int_{\frac{j+1}{2}-\delta}^{\frac{j+1}{2}+\delta} (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) e^{i\tau\xi - (2\beta-\pi)(\xi - \frac{j+1}{2})} \right. \\ &\quad \left. \times e^{-(2k+1)\pi(\xi+ih)} d\xi \right) \lambda^j \\ &= e^{-(2\beta-\pi)ih} \sum_{j \geq 0} \sum_{k \geq 1} \left(e^{i\tau \frac{j+1}{2} - (2k+1)\pi \frac{j+1}{2}} \int_{-\delta}^{\delta} (\xi + ih) \left(\xi + ih + \frac{j+1}{2} \right) e^{i\tau\xi - (2\beta-\pi)\xi} \right. \\ &\quad \left. \times e^{-(2k+1)\pi(\xi+ih)} d\xi \right) \lambda^j \\ &= e^{-(2\beta-\pi)ih} \sum_{k \geq 1} \left(\frac{1}{e^{[(2k+1)\pi-i\tau]/2} - \lambda} \int_{-\delta}^{\delta} (\xi + ih)^2 e^{i\tau\xi - (2\beta-\pi)\xi} e^{-(2k+1)\pi(\xi+ih)} d\xi \right. \\ &\quad \left. + \frac{e^{[(2k+1)\pi-i\tau]/2}}{(e^{[(2k+1)\pi-i\tau]/2} - \lambda)^2} \int_{-\delta}^{\delta} (\xi + ih) e^{i\tau\xi - (2\beta-\pi)\xi} e^{-(2k+1)\pi(\xi+ih)} d\xi \right) \\ &= e^{-(2\beta-\pi)ih} \sum_{k \geq 1} \left(\frac{h_1^{(k)}(\tau)}{e^{[(2k+1)\pi-i\tau]/2} - \lambda} + \frac{h_2^{(k)}(\tau) e^{[(2k+1)\pi-i\tau]/2}}{(e^{[(2k+1)\pi-i\tau]/2} - \lambda)^2} \right), \end{aligned}$$

where $h_1^{(k)}, h_2^{(k)}$ are entire functions of exponential type, with bounds uniform in k . Notice that, when summing in j , the series converges for $|\lambda| < e^{\text{Im } \tau + (2k+1)\pi}$.

Claim 1. We now claim that the above sum converges to a function $E_1(\tau, \lambda)$, holomorphic on the domain

$$\mathcal{D}_{\infty, 2\pi} = \{(\tau, \lambda) : |\text{Im } \tau - \log |\lambda|^2| < 2\pi, |\lambda| > 0\},$$

that is of exponential type in τ uniformly in λ when λ varies in a compact set.

For, by the observation above, the functions to be summed are all holomorphic in $\mathcal{D}_{\infty, 2\pi}$. Next, for fixed $M > 0$ we can select k_0 large enough so that for all $k \geq k_0$, when $(\tau, \lambda) \in \mathcal{D}_{\infty, 2\pi}$, with $|\operatorname{Im} \tau| \leq M$ and $|\lambda| \leq e^M$, we have that

$$\left| e^{[(2k+1)\pi - i\tau]/2} - \lambda \right| \geq e^{[(2k+1)\pi - M]/2} - e^M \geq \frac{1}{2} e^{\pi k}.$$

Therefore

$$\left| \frac{h_1^{(k)}(\tau)}{e^{[(2k+1)\pi - i\tau]/2} - \lambda} \right| \leq c e^{-\pi k},$$

and similarly

$$\left| \frac{h_2^{(k)}(\tau) e^{[(2k+1)\pi - i\tau]/2}}{(e^{[(2k+1)\pi - i\tau]/2} - \lambda)^2} \right| \leq c e^{-\pi k} \frac{e^{[(2k+1)\pi + M]/2}}{e^{[(2k+1)\pi - M]/2} - e^M} \leq c e^{-\pi k},$$

so that the two series above converge uniformly in the fixed compact set. This proves the claim.

10.3. The Cases of $\sum_{j < 0} \mathcal{E}_1^* \lambda^j$ and $\sum_{j < 0} \mathcal{E}_2^* \lambda^j$. Now we turn to the sums over negative indices j . Clearly it suffices to consider the sum for $j \leq j_0 < -1$. On the relevant interval of integration, we can then write $m_1(\xi) = \sum_{k=1}^{+\infty} e^{2k\pi(\xi + ih)}$, and the series converges uniformly there. We have

$$\begin{aligned} \sum_{j \leq j_0} \mathcal{E}_1^* \lambda^j &= \sum_{j \leq j_0} e^{2\beta i h} \left(\int_{\frac{j+1}{2} - \delta}^{\frac{j+1}{2} + \delta} (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) e^{i\tau\xi + \pi\xi + (2\beta - \pi)(\xi - \frac{j+1}{2})} m_1(\xi) d\xi \right) \lambda^j \\ &= \sum_{j \leq j_0} \sum_{k \geq 1} e^{2\beta i h} \left(\int_{\frac{j+1}{2} - \delta}^{\frac{j+1}{2} + \delta} (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) e^{i\tau\xi + \pi\xi + (2\beta - \pi)(\xi - \frac{j+1}{2})} \right. \\ &\quad \left. \times e^{2k\pi(\xi + ih)} d\xi \right) \lambda^j \\ &= e^{2\beta i h} \sum_{k \geq 1} \sum_{j \leq j_0} e^{[i\tau + (2k+1)\pi] \frac{j+1}{2}} \left(\int_{-\delta}^{\delta} (\xi + ih)^2 e^{i\tau\xi + (2k+1)\pi\xi + (2\beta - \pi)\xi} e^{2k i h} d\xi \right. \\ &\quad \left. + \frac{j+1}{2} \int_{-\delta}^{\delta} (\xi + ih) e^{i\tau\xi + (2k+1)\pi\xi + (2\beta - \pi)\xi} e^{2k i h} d\xi \right) \lambda^j. \end{aligned} \quad (47)$$

Again we must sum in j . Since

$$\sum_{j \leq j_0} e^{\gamma \frac{j+1}{2}} \lambda^j = \frac{(e^{\gamma/2} \lambda)^{j_0+1}}{\lambda - e^{-\gamma/2}} \quad (48)$$

and

$$\sum_{j \leq j_0} \frac{j+1}{2} e^{\gamma \frac{j+1}{2}} \lambda^j = - \frac{(e^{\gamma/2} \lambda)^{j_0+1} ((j_0+1)(e^{-\gamma/2} - \lambda) + e^{-\gamma/2})}{2(\lambda - e^{-\gamma/2})^2}, \quad (49)$$

we have, when $|e^{\gamma/2} \lambda| > 1$, i.e. when $|\lambda| > e^{-\gamma/2}$, that

$$\begin{aligned} \sum_{j \leq j_0} \mathcal{E}_1^* \lambda^j &= e^{2\beta i h} \sum_{k \geq 1} \left(e^{2k i h} \frac{(\lambda - e^{[i\tau + (2k+1)\pi]/2})^{j_0+1}}{\lambda - e^{-[i\tau + (2k+1)\pi]}} \int_{-\delta}^0 (\xi + i h)^2 e^{i\tau \xi + (2k+1)\pi \xi + (2\beta - \pi)\xi} d\xi \right. \\ &\quad \left. + e^{2k i h} \frac{(e^{[i\tau + (2k+1)\pi]/2} \lambda)^{j_0+1} ((j_0+1)(e^{-[i\tau + (2k+1)\pi]/2} - \lambda) + e^{-[i\tau + (2k+1)\pi]/2})}{2(\lambda - e^{-[i\tau + (2k+1)\pi]/2})^2} \right. \\ &\quad \left. \times \int_{-\delta}^0 (\xi + i h) e^{i\tau \xi + (2k+1)\pi \xi + (2\beta - \pi)\xi} d\xi \right) \\ &= \sum_{k \geq 1} \left(\frac{h_3^{(k)}(\tau, \lambda)}{\lambda - e^{-[i\tau + (2k+1)\pi]/2}} + \frac{h_4^{(k)}(\tau, \lambda)}{2(\lambda - e^{-[i\tau + (2k+1)\pi]/2})^2} \right), \end{aligned} \quad (50)$$

where $h_3^{(k)}, h_4^{(k)}$ are entire functions that are of exponential type in τ , uniformly in k and λ , for λ varying in any compact set. Notice that the above series in j converge when $|\lambda| > e^{[\text{Im } \tau - (2k+1)]/2}$ hence, in particular, when $(\tau, \lambda) \in \mathcal{D}_{\infty, 2\pi}$.

Arguing as in Claim 1 (Subsection 10.2), we see that the above sum converges to a function $E_2(\tau, \lambda)$, holomorphic on $\mathcal{D}_{\infty, 2\pi}$, which is of exponential type in τ , uniformly in λ for λ varying in any compact sets. For, having fixed an $M > 0$, we can select k_0 such that if $k \geq k_0$, if $|\text{Im } \tau| \leq M$ and $|\lambda| \leq e^M$,

$$\left| \frac{1}{(e^{[(2k+1)\pi - i\tau]/2} - \lambda)^2} \right| \leq c e^{-\pi k} \quad \text{and} \quad |h_3^{(k)}(\tau, \lambda)|, |h_4^{(k)}(\tau, \lambda)| \leq C.$$

Now we turn to the sum

$$\begin{aligned} \sum_{j \leq -1} \mathcal{E}_2^* \lambda^j &= \sum_{j \leq -1} e^{2\beta i h} \int_{-\delta}^{\delta} (\xi + i h) \left(\xi + i h - \frac{j+1}{2} \right) e^{i\tau \xi - \pi \xi - (2\beta - \pi)(\xi - \frac{j+1}{2})} m_1(\xi) d\xi \lambda^j \\ &= e^{2\beta i h} \sum_{j \leq -1} e^{(2\beta - \pi) \frac{j+1}{2}} \left(\int_{-\delta}^{\delta} (\xi + i h)^2 e^{(i\tau - 2\beta - \pi)\xi} m_1(\xi) d\xi \right. \\ &\quad \left. - \frac{j+1}{2} \int_{-\delta}^{\delta} (\xi + i h) e^{(i\tau - 2\beta - \pi)\xi} m_1(\xi) d\xi \right) \lambda^j. \end{aligned}$$

Using formulas (48) and (49) with $j_0 = -1$ we see that, for $|\lambda| > e^{-(\beta - \pi/2)}$,

$$\sum_{j < 0} \mathcal{E}_2^* \lambda^j = \frac{g_1(\tau)}{\lambda - e^{-(\beta - \pi/2)}} + \frac{g_2(\tau)}{(\lambda - e^{-(\beta - \pi/2)})^2} = \frac{g_1(\tau)(\lambda - e^{-(\beta - \pi/2)}) + g_2(\tau)}{(\lambda - e^{-(\beta - \pi/2)})^2},$$

where g_1, g_2 are entire functions of exponential type, that is, they decay exponentially, together with all their derivatives, in every closed horizontal strip.

According to the decompositions (44) and (45), in order to conclude the analysis of the term $\sum_{j \in \mathbb{Z}} E_j^{(1)} \lambda^j$, it remains to consider the sums of the terms involving I through III and I^* through III^* , that is,

$$\sum_{j \geq 0} (I - II + III) \lambda^j + \sum_{j < 0} (I^* - II^* + III^*) \lambda^j.$$

10.4. The Case of $\sum_{j \geq 0} (I + III) \lambda^j + \sum_{j < 0} (I^* + III^*) \lambda^j$. Notice that, when $\operatorname{Re} \tau \geq 0$, since h is fixed with $0 < h < 1/2$, the series $\sum_{k \geq 1} e^{-2k\pi \operatorname{sgn}(\xi)(\xi + ih)}$ converging to $m_1(\xi)$ has partial sums that are uniformly bounded. Thus, by the Lebesgue dominated convergence theorem, we can interchange the order of summation and integration. Recalling the definitions of σ and $\sigma_{R,S}$, (32) and (40), we then write

$$\begin{aligned} & \sum_{j \geq 0} (I + III) \lambda^j \\ &= \sum_{j \geq 0} \lambda^j \sum_{k \geq 1} \left(\int_{-\infty}^{-\delta} + \int_{\frac{j+1}{2} + \delta}^{+\infty} \right) \sigma(\xi) (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) \\ & \quad \times e^{i\xi\tau} e^{-\pi|\xi|} e^{-(2\beta-\pi)|\xi - \frac{j+1}{2}|} e^{-2\pi k \operatorname{sgn}(\xi)(\xi + ih)} d\xi \\ &= \sum_{k=1}^{+\infty} \sum_{j \geq 0} \lambda^j \left(\int_{-\infty}^{-\delta} + \int_{\frac{j+1}{2} + \delta}^{+\infty} \right) \sigma_{(2\beta-\pi), (2k+1)\pi}(\xi) (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) \\ & \quad \times e^{i\xi\tau} e^{-(2k+1)\pi|\xi|} e^{-(2\beta-\pi)|\xi - \frac{j+1}{2}|} d\xi \\ &= \sum_{k=1}^{+\infty} \sum_{j \geq 0} \lambda^j \mathcal{I}_j^{(k)}(\tau). \end{aligned} \tag{51}$$

Analogous arguments apply when $j < 0$ to give

$$\begin{aligned} & \sum_{j < 0} (I^* + III^*) \lambda^j \\ &= \sum_{j < 0} \lambda^j \sum_{k \geq 1} \left(\int_{-\infty}^{\frac{j+1}{2} - \delta} + \int_{\delta}^{+\infty} \right) \sigma(\xi) (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) \\ & \quad \times e^{i\xi\tau} e^{-\pi|\xi|} e^{-(2\beta-\pi)|\xi - \frac{j+1}{2}|} e^{-2\pi k \operatorname{sgn}(\xi)(\xi + ih)} d\xi \\ &= \sum_{k=1}^{+\infty} \sum_{j < 0} \lambda^j \left(\int_{-\infty}^{\frac{j+1}{2} - \delta} + \int_{\delta}^{+\infty} \right) \sigma_{(2\beta-\pi), (2k+1)\pi}(\xi) (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) \\ & \quad \times e^{i\xi\tau} e^{-(2k+1)\pi|\xi|} e^{-(2\beta-\pi)|\xi - \frac{j+1}{2}|} d\xi \\ &= \sum_{k=1}^{+\infty} \sum_{j < 0} \lambda^j \mathcal{I}_j^{(k)}(\tau). \end{aligned} \tag{52}$$

In order to calculate the sum $\sum_{j \in \mathbb{Z}} \mathcal{I}_j^{(k)}(\tau) \lambda^j$, we apply Lemma 9.4 and the identity (43) with $R = 2\beta - \pi$, $S = (2k + 1)\pi$ and $\delta > 0$ a fixed parameter. The double sum above equals

$$\begin{aligned} \sum_{k \geq 1} \frac{e^{ih(2\beta - \pi + S) - \delta[i\tau + 2\beta - \pi + S]}}{(i\tau + 2\beta - \pi + S)^2} & \left[\frac{e^{(2\beta - \pi)/2}}{2(e^{(2\beta - \pi)/2} - \lambda)^2} (1 + \psi_1^+(\tau, \lambda)) \right. \\ & + \frac{e^{-[i\tau + S]/2}}{2(\lambda - e^{-[i\tau + S]/2})^2} (1 + \psi_2^-(\tau, \lambda)) + \frac{e^{-[i\tau + S]/2} \psi_4^+(\tau)}{2(e^{(2\beta - \pi)/2} - \lambda)(e^{-[i\tau + S]/2} - \lambda)} \Big] \\ & + \frac{e^{-ih(2\beta - \pi + S) - \delta[2\beta - \pi + S - i\tau]}}{(i\tau - (2\beta - \pi) - S)^2} \left[\frac{e^{[S - i\tau]/2}}{2(e^{[S - i\tau]/2} - \lambda)^2} (1 + \psi_2^+(\tau, \lambda)) \right. \\ & + \frac{e^{-(2\beta - \pi)/2}}{2(\lambda - e^{-(2\beta - \pi)/2})^2} (1 + \psi_1^-(\tau, \lambda)) + \frac{e^{[S - i\tau]/2} \psi_4^-(\tau)}{2(e^{-(2\beta - \pi)/2} - \lambda)(e^{[S - i\tau]/2} - \lambda)} \Big], \end{aligned} \quad (53)$$

where $S = (2k + 1)\pi$ and $\psi_j^\pm \equiv \psi_{(2\beta - \pi, (2k + 1)\pi), j}^\pm$ depend also on the parameter δ and are defined in (72), (81), and (68). Here we clearly need to stress the dependence on S .

We are going to show that the functions depending on k , that is on S , can be summed, and their sums are functions of (τ, λ) , holomorphic in a neighborhood of $\overline{\mathcal{D}}$, bounded (together with their derivatives) as $|\operatorname{Re} \tau| \rightarrow +\infty$.

We let (τ, λ) vary in the closure of the domain

$$\mathcal{D}_{2\beta, 3\pi/2} = \{ |\operatorname{Im} \tau - \log |\lambda|^2| < 3\pi/2, \ e^{-\beta} < |\lambda| < e^\beta \},$$

a domain that contains the closure of the domain \mathcal{D} . Using Lemma 9.2, we now obtain that

$$\begin{aligned} \left| \frac{e^{ih(2\beta - \pi + S) - \delta[i\tau + 2\beta - \pi + S]}}{(i\tau + 2\beta - \pi + S)^2} \right| \cdot |e^{(2\beta - \pi)/2} (1 + \psi_1^+(\tau, \lambda))| & \leq C(|\operatorname{Re} \tau| + S) \frac{e^{-\delta[i\tau + R + S]}}{|i\tau + R + S|^2} \\ & \leq C e^{-\delta S} \end{aligned} \quad (54)$$

as $S \rightarrow +\infty$, uniformly in $(\tau, \lambda) \in \mathcal{D}_{2\beta, 3\pi/2}$.

Next notice that the functions $\frac{e^{-[i\tau + S]/2}}{2(\lambda - e^{-[i\tau + S]/2})^2}$ are bounded for $(\tau, \lambda) \in \mathcal{D}_{2\beta, 3\pi/2}$, uniformly in $S = (2k + 1)\pi$. Then

$$\begin{aligned} \left| \frac{e^{ih(2\beta - \pi + S) - \delta[i\tau + 2\beta - \pi + S]}}{(i\tau + 2\beta - \pi + S)^2} \cdot \frac{e^{-[i\tau + S]/2}}{2(\lambda - e^{-[i\tau + S]/2})^2} (1 + \psi_2^-(\tau, \lambda)) \right| \\ \leq C e^{-\delta S} \frac{|\operatorname{Re} \tau| + S}{|i\tau + (2\beta - \pi) + S|^2} \\ \leq C e^{-\delta S} \end{aligned} \quad (55)$$

as $S \rightarrow +\infty$, uniformly in $(\tau, \lambda) \in \mathcal{D}_{2\beta, 3\pi/2}$.

Arguing as above, we see that

$$\left| \frac{e^{ih(2\beta-\pi+S)-\delta[i\tau+2\beta-\pi+S]}}{(i\tau+2\beta-\pi+S)^2} \cdot \frac{e^{-[i\tau+S]/2}\psi_4^+(\tau)}{2(e^{-[i\tau+S]/2}-\lambda)} \right| \leq Ce^{-\delta S} \frac{1}{|i\tau+(2\beta-\pi)+S|^2} \leq Ce^{-\delta S}, \quad (56)$$

again, as $S \rightarrow +\infty$, uniformly in $(\tau, \lambda) \in \mathcal{D}_{2\beta, 3\pi/2}$.

Next we notice that the functions $\frac{e^{[S-i\tau]}}{2(e^{[S-i\tau]/2}-\lambda)^2}$ are bounded for $(\tau, \lambda) \in \mathcal{D}_{2\beta, 3\pi/2}$ uniformly in $S = (2k+1)\pi$. For it suffices to notice that

$$|\lambda e^{-[S-i\tau]/2}|^2 \leq e^{-3\pi/2}$$

for all $(\tau, \lambda) \in \mathcal{D}_{2\beta, 3\pi/2}$ and all $k \geq 1$.

Then, using Lemma 9.2 again, we have

$$\begin{aligned} & \left| \frac{e^{-ih(2\beta-\pi+S)-\delta[2\beta-\pi+S-i\tau]}}{(i\tau-(2\beta-\pi)-S)^2} \cdot \frac{e^{[S-i\tau]/2}}{2(e^{[S-i\tau]/2}-\lambda)^2} (1 + \psi_2^+(\tau, \lambda)) \right| \\ & \leq Ce^{-\delta S} \frac{(|\operatorname{Re} \tau| + S)e^{-S/2}}{|i\tau-(2\beta-\pi)-S|^2} \\ & \leq Ce^{-(1+\delta)S}, \end{aligned} \quad (57)$$

$$\begin{aligned} & \left| \frac{e^{-ih(2\beta-\pi+S)-\delta[2\beta-\pi+S-i\tau]}}{(i\tau-(2\beta-\pi)-S)^2} (1 + \psi_1^-(\tau, \lambda)) \right| \leq Ce^{-\delta S} \frac{|\operatorname{Re} \tau| + S}{|i\tau-(2\beta-\pi)-S|^2} \\ & \leq Ce^{-\delta S}, \end{aligned} \quad (58)$$

and

$$\begin{aligned} & \left| \frac{e^{-ih(2\beta-\pi+S)-\delta[2\beta-\pi+S-i\tau]}}{(i\tau-(2\beta-\pi)-S)^2} \cdot \frac{e^{[S-i\tau]/2}\psi_4^-(\tau)}{2(e^{[S-i\tau]/2}-\lambda)} \right| \leq C \frac{e^{-\delta S}}{|i\tau-(2\beta-\pi)-S|^2} \\ & \leq Ce^{-\delta S}. \end{aligned} \quad (59)$$

Next we have

10.5. The Case of $\sum_{j \geq 0} II\lambda^j + \sum_{j < 0} II^*\lambda^j$. In this case we will use Lemmas 9.3 and 9.5, and summation by parts. We begin with the case $j \geq 0$. Notice that, for $\xi \geq \delta$, for any positive integer N , $m_1(\xi) = \sum_{k \geq 1}^N e^{-2k\pi(\xi+ih)} + e^{-2N\pi(\xi+ih)}m_1(\xi)$.

Therefore

$$\begin{aligned}
& \sum_{j \geq 0} II \lambda^j \\
&= e^{ih(2\beta-2\pi)} \sum_{j \geq 0} e^{-(2\beta-\pi)\frac{j+1}{2}} \lambda^j \int_{\delta}^{\frac{j+1}{2}-\delta} [\xi^2 + b\xi + c] e^{(i\tau+(2\beta-\pi)-\pi)\xi} m_1(\xi) d\xi \\
&= e^{ih(2\beta-2\pi)} \sum_{j \geq 0} e^{-(2\beta-\pi)\frac{j+1}{2}} \lambda^j \left(\sum_{k=1}^N e^{-2\pi i k h} \int_{\delta}^{\frac{j+1}{2}-\delta} [\xi^2 + b\xi + c] e^{(i\tau+(2\beta-\pi)-(2k+1)\pi)\xi} d\xi \right. \\
&\quad \left. + \int_{\delta}^{\frac{j+1}{2}-\delta} [\xi^2 + b\xi + c] e^{(i\tau+(2\beta-\pi)-(2N+1)\pi)\xi} m_1(\xi) d\xi \right) \\
&\equiv A + B .
\end{aligned}$$

Now we apply Lemma 9.3 with $R = 2\beta - \pi$ and $S = (2k + 1)\pi$, $k = 1, \dots, N$. Then we see that there exists a function $\Psi_1(\tau, \lambda)$ holomorphic on the closure of $\mathcal{D}_{2\beta, 3\pi/2}$, bounded (together with all their derivatives) as $|\operatorname{Re} \tau| \rightarrow +\infty$, such that

$$A = \frac{\Psi_1(\tau, \lambda)}{(e^{(\beta-\pi/2)} - \lambda)^2} .$$

In order to evaluate B we use the summation by parts formula

$$\sum_{j=0}^N a_j b_j = \sum_{k=0}^{N-1} s_k (b_{k+1} - b_k) + s_N b_N - s_0 b_0 , \quad (60)$$

where $s_k = \sum_{j=0}^k a_j$, $a_j = e^{-(2\beta-\pi)\frac{j+1}{2}} \lambda^j$ and

$$b_j = \int_{\delta}^{\frac{j+1}{2}-\delta} [\xi^2 + b\xi + c] e^{(i\tau+(2\beta-\pi)-(2N+1)\pi)\xi} m_1(\xi) d\xi .$$

Notice that

$$s_k = \sum_{j=0}^k e^{-(2\beta-\pi)\frac{j+1}{2}} \lambda^j = \frac{1 - \lambda^{k+1} e^{-k(\beta-\pi/2)}}{e^{(\beta-\pi/2)} - \lambda} ,$$

and that

$$\begin{aligned}
b_{k+1} - b_k &= \left(\int_{\delta}^{\frac{k+3}{2}-\delta} - \int_{\delta}^{\frac{k+1}{2}-\delta} \right) [\xi^2 + b\xi + c] e^{(i\tau+(2\beta-\pi)-(2N+1)\pi)\xi} m_1(\xi) d\xi \\
&= \int_{1-\delta}^{2-\delta} \left[\left(\xi + \frac{k-1}{2} \right)^2 + b \left(\xi + \frac{k-1}{2} \right) + c \right] e^{(i\tau+(2\beta-\pi)-(2N+1)\pi)(\xi+\frac{k-1}{2})} \\
&\quad \times m_1 \left(\xi + \frac{k-1}{2} \right) d\xi \\
&= e^{(i\tau+(2\beta-\pi)-(2N+1)\pi)(\frac{k-1}{2})} \int_{1-\delta}^{2-\delta} \left[\left(\xi + \frac{k-1}{2} \right)^2 + b \left(\xi + \frac{k-1}{2} \right) + c \right] \\
&\quad \times e^{(i\tau+(2\beta-\pi)-(2N+1)\pi)(\xi+\frac{k-1}{2})} m_1 \left(\xi + \frac{k-1}{2} \right) d\xi .
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{k=0}^{N-1} s_k (b_{k+1} - b_k) \\
&= \sum_{k=0}^{N-1} \frac{1 - \lambda^k e^{-k(\beta-\pi/2)}}{e^{(\beta-\pi/2)} - \lambda} e^{(i\tau+(2\beta-\pi)-(2N+1)\pi)(\frac{k-1}{2})} (c_2 k^2 + c_1 k + c_0) , \quad (61)
\end{aligned}$$

where $c_j = c_j(\tau, k)$ are entire functions of τ , of exponential type, with bounds uniform in k .

Now it is not hard to see that, as we let $N \rightarrow +\infty$, the term on the righthand side of equation (61) above converges to a function $\Psi_2(\tau, \lambda)$, holomorphic in a neighborhood of the closure of $\mathcal{D}_{2\beta, 3\pi/2}$, times $1/(e^{(\beta-\pi/2)} - \lambda)$.

The sum for $j < 0$ is treated similarly, and we do not repeat the argument here. Hence (relabeling the functions) we obtain that there exist functions Ψ_1, Ψ_2 , holomorphic on the closure of $\mathcal{D}_{2\beta, 3\pi/2}$, bounded (together with all their derivatives) as $|\operatorname{Re} \tau| \rightarrow +\infty$, such that the sum $\sum_{j \geq 0} (II) \lambda^j + \sum_{j < 0} (II^*) \lambda^j$, relative to the error term E_1 , equals

$$\frac{\Psi_1(\tau, \lambda)}{(e^{R/2} - \lambda)^2} + \frac{\Psi_2(\tau, \lambda)}{(e^{-R/2} - \lambda)^2} .$$

This proves the statement for the error term E_1 .

10.6. The Sum of the $E_j^{(2)}$. The analysis of the sum $\sum_{j \in \mathbb{Z}} E_j^{(2)} \lambda^j$ is quite similar to the previous case of $E_j^{(1)}$. In fact, the change of variables $\xi' = \xi + \frac{j+1}{2}$ and rescaling allow one to apply the previous proof to this case.

10.7. **The Sum of the $E_j^{(3)}$.** Finally, we consider the sum $\sum_{j \in \mathbb{Z}} E_j^{(3)} \lambda^j$. Recall that

$$E_j^{(3)}(\tau) = \frac{2}{\pi} e^{-h \operatorname{sgn}(\operatorname{Re} \tau) \tau} \int_{-\infty}^{+\infty} \sigma(\xi) e^{i\tau \xi} \frac{(\xi + ih)(\xi + ih - \frac{j+1}{2})}{e^{\pi|\xi|} e^{(2\beta-\pi)|\xi - \frac{j+1}{2}|}} \\ \times \frac{e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}}{1 - e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}} \cdot \frac{e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}}{1 - e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}} d\xi.$$

Again, we begin with the case $\operatorname{Re} \tau \geq 0$.

We divide the integral above as in (41). We write

$$\int_{-\infty}^{+\infty} \sigma(\xi) e^{i\tau \xi} m_1(\xi) \tilde{m}(\xi) \frac{(\xi + ih)(\xi + ih - \frac{j+1}{2})}{e^{\pi|\xi|} e^{(2\beta-\pi)|\xi - \frac{j+1}{2}|}} d\xi \\ = \left(\int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\frac{j+1}{2}-\delta} + \int_{\frac{j+1}{2}-\delta}^{\frac{j+1}{2}+\delta} + \int_{\frac{j+1}{2}+\delta}^{+\infty} \right) \\ \times \sigma(\xi) m_1(\xi) \tilde{m}(\xi) \frac{(\xi + ih)(\xi + ih - \frac{j+1}{2}) e^{i\tau \xi}}{e^{\pi|\xi|} e^{(2\beta-\pi)|\xi - \frac{j+1}{2}|}} d\xi \\ \equiv I + \mathcal{E}_1 - II + \mathcal{E}_2 + III,$$

where, as before,

$$m_1(\xi) = \frac{e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}}{1 - e^{-2\operatorname{sgn}(\xi)\pi(\xi+ih)}} \quad \text{and} \quad \tilde{m}(\xi) = \frac{e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}}{1 - e^{-2\operatorname{sgn}(\xi - \frac{j+1}{2})(2\beta-\pi)(\xi - \frac{j+1}{2} + ih)}}.$$

We first consider the sum involving \mathcal{E}_1 :

$$\sum_{j \geq 0} \mathcal{E}_1 \lambda^j \\ = e^{2\beta ih} \sum_{j \geq 0} \left(\int_{-\delta}^{\delta} m_1(\xi) (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) e^{i\tau \xi + \pi \xi + (2\beta-\pi)(\xi - \frac{j+1}{2})} \tilde{m}(\xi) d\xi \right) \lambda^j \\ = e^{2\beta ih} \sum_{j \geq 0} \left(\sum_{k \geq 1} e^{2k(2\beta-\pi)ih} \int_{-\delta}^{\delta} m_1(\xi) (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) \right. \\ \left. \times e^{i\tau \xi + \pi \xi + (2\beta-\pi)(2k+1)(\xi - \frac{j+1}{2})} d\xi \right) \lambda^j$$

$$\begin{aligned}
&= e^{2\beta ih} \sum_{k \geq 1} e^{2k(2\beta-\pi)ih} \sum_{j \geq 0} e^{-(2\beta-\pi)(2k+1)\frac{j+1}{2}} \left(\int_{-\delta}^{\delta} m_1(\xi)(\xi+h)^2 e^{i\tau\xi+\pi\xi+(2\beta-\pi)(2k+1)\xi} d\xi \right. \\
&\quad \left. - \frac{j+1}{2} \int_{-\delta}^{\delta} m_1(\xi)(\xi+h) e^{i\tau\xi+\pi\xi+(2\beta-\pi)(2k+1)\xi} d\xi \right) \lambda^j \\
&= e^{2\beta ih} \sum_{k \geq 1} e^{2k(2\beta-\pi)ih} \left(\frac{1}{e^{((\beta-\pi/2))(2k+1)} - \lambda} \int_{-\delta}^{\delta} m_1(\xi)(\xi+h)^2 e^{i\tau\xi+\pi\xi+(2\beta-\pi)(2k+1)\xi} d\xi \right. \\
&\quad \left. - \frac{e^{((\beta-\pi/2))(2k+1)}}{2(e^{((\beta-\pi/2))(2k+1)} - \lambda)^2} \int_{-\delta}^{\delta} m_1(\xi)(\xi+h) e^{i\tau\xi+\pi\xi+(2\beta-\pi)(2k+1)\xi} d\xi \right).
\end{aligned}$$

The last series converge when $|\lambda| < e^{(\beta-\pi/2)(2k+1)}$.

Now we fix $0 < \delta < 1$. Then the series

$$\sum_{k \geq 1} e^{2k(2\beta-\pi)ih} \frac{e^{(2\beta-\pi)(2k+1)\xi}}{e^{((\beta-\pi/2))(2k+1)} - \lambda}, \quad \text{and} \quad \sum_{k \geq 1} e^{2k(2\beta-\pi)ih} \frac{e^{(2\beta-\pi)(2k+1)(\xi+1)}}{2(e^{((\beta-\pi/2))(2k+1)} - \lambda)^2}$$

converge uniformly for $\xi \in [-\delta, \delta]$ to functions of ξ and λ , C^∞ in ξ and holomorphic in λ for $|\lambda| < e^{3(\beta-\pi/2)}$. Therefore we obtain that $\sum_{j \geq 0} \mathcal{E}_1 \lambda^j$ converges to a function $E(\tau, \lambda)$, holomorphic for $|\lambda| < e^{3(\beta-\pi/2)}$, entire in τ and of exponential decay in any closed horizontal strip, uniformly in λ for $|\lambda| \leq C < e^{3(\beta-\pi/2)}$.

We now consider the sum involving \mathcal{E}_2 . We have

$$\begin{aligned}
&\sum_{j \geq 0} \mathcal{E}_2 \lambda^j \\
&= e^{-2\beta ih} \sum_{j \geq 0} \int_{\frac{j+1}{2}-\delta}^{\frac{j+1}{2}+\delta} m_1(\xi) \tilde{m}(\xi) (\xi+ih) \left(\xi+ih - \frac{j+1}{2} \right) e^{i\tau\xi-\pi\xi-(2\beta-\pi)(\xi-\frac{j+1}{2})} d\xi \lambda^j \\
&= e^{-2\beta ih} \sum_{j \geq 0} \sum_{k \geq 1} e^{-2k\pi ih} \int_{\frac{j+1}{2}-\delta}^{\frac{j+1}{2}+\delta} \tilde{m}(\xi) (\xi+ih) \left(\xi+ih - \frac{j+1}{2} \right) e^{i\tau\xi-(2k+1)\pi\xi-(2\beta-\pi)(\xi-\frac{j+1}{2})} d\xi \lambda^j \\
&= e^{-(2\beta-\pi)ih} \sum_{j \geq 0} \sum_{k \geq 1} e^{-(2k+1)\pi ih} e^{(i\tau-(2k+1)\pi)\frac{j+1}{2}} \left(\int_{-\delta}^{\delta} m_2(\xi)(\xi+ih)^2 e^{i\tau\xi-(2k+1)\pi\xi-(2\beta-\pi)\xi} d\xi \right. \\
&\quad \left. + \frac{j+1}{2} \int_{-\delta}^{\delta} m_2(\xi)(\xi+ih) e^{i\tau\xi-(2k+1)\pi\xi-(2\beta-\pi)\xi} d\xi \right) \lambda^j \\
&= e^{-(2\beta-\pi)ih} \sum_{k \geq 1} e^{-(2k+1)\pi ih} \left(\frac{1}{e^{[(2k+1)\pi-i\tau]/2} - \lambda} \int_{-\delta}^{\delta} m_2(\xi)(\xi+ih)^2 e^{i\tau\xi-(2k+1)\pi\xi-(2\beta-\pi)\xi} d\xi \right. \\
&\quad \left. + \left(\frac{e^{[(2k+1)\pi-i\tau]/2}}{2(e^{[(2k+1)\pi-i\tau]/2} - \lambda)^2} \int_{-\delta}^{\delta} m_2(\xi)(\xi+ih) e^{i\tau\xi-(2k+1)\pi\xi-(2\beta-\pi)\xi} d\xi \right) \right),
\end{aligned}$$

where we have set $\tilde{m}(\xi + \frac{j+1}{2}) \equiv m_2(\xi)$, since it does not depend on j . When summing in j , the series converges for $|\lambda| < e^{\text{Im } \tau + (2k+1)\pi}$.

The term on the righthand side above equals

$$e^{-(2\beta-\pi)ih} \sum_{k \geq 1} \left(\frac{h_1^{(k)}(\tau)}{e^{[(2k+1)\pi-i\tau]/2} - \lambda} + \frac{h_2^{(k)}(\tau)e^{[(2k+1)\pi-i\tau]/2}}{(e^{[(2k+1)\pi-i\tau]/2} - \lambda)^2} \right),$$

where $h_1^{(k)}, h_2^{(k)}$ are entire functions of exponential type, with bounds uniform in k . Arguing as in Claim 1 (Subsection 10.2), we obtain that the above sum converges to a function $E_2(\tau, \lambda)$ holomorphic in the domain $\mathcal{D}_{\infty, 2\pi}$ that is of exponential type in τ , uniformly in λ , when λ varies in a compact set.

Now we consider the sums for the other two error terms, \mathcal{E}_1^* and \mathcal{E}_2^* . These sums involve j when $j < 0$.

We have

$$\begin{aligned} \sum_{j < 0} \mathcal{E}_1^* \lambda^j &= \sum_{j < 0} e^{2\beta ih} \left(\int_{\frac{j+1}{2}-\delta}^{\frac{j+1}{2}+\delta} m_1(\xi) \tilde{m}(\xi) (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) \right. \\ &\quad \left. \times e^{i\tau\xi + \pi\xi + (2\beta-\pi)(\xi - \frac{j+1}{2})} d\xi \right) \lambda^j, \end{aligned}$$

and $m_1(\xi) \equiv \tilde{m}(\xi + \frac{j+1}{2})$ does not depend on j . We can argue as in (47) and (50) to obtain that the above sum converges to a function $E_3(\tau, \lambda)$, holomorphic on the closure of $\mathcal{D}_{\infty, 2\pi}$, which is of exponential type in τ , uniformly in λ for λ varying in any compact set.

Next, expanding $\tilde{m}(\xi)$ in Taylor series, we see that

$$\begin{aligned} &\sum_{j \leq -1} \mathcal{E}_2^* \lambda^j \\ &= e^{2\beta ih} \sum_{j \leq -1} \int_{-\delta}^{\delta} (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) m_1(\xi) \tilde{m}(\xi) e^{i\tau\xi - \pi\xi - (2\beta-\pi)(\xi - \frac{j+1}{2})} d\xi \lambda^j \\ &= \sum_{k \geq 1} e^{-(2k-1)(2\beta-\pi)ih} \sum_{j \leq -1} e^{(2\beta-\pi)(2k+1)\frac{j+1}{2}} \left(\int_{-\delta}^{\delta} (\xi + ih)^2 m_1(\xi) e^{i\tau\xi - \pi\xi - (2\beta-\pi)(2k+1)\xi} d\xi \right. \\ &\quad \left. - \frac{j+1}{2} \int_{-\delta}^{\delta} (\xi + ih) m_1(\xi) e^{i\tau\xi - \pi\xi - (2\beta-\pi)(2k+1)\xi} d\xi \right) \lambda^j \end{aligned}$$

$$\begin{aligned}
&= \sum_{k \geq 1} e^{-(2k-1)(2\beta-\pi)ih} \frac{1}{\lambda - e^{-(2\beta-\pi)(2k+1)/2}} \left(\int_{-\delta}^{\delta} (\xi + ih)^2 m_1(\xi) e^{i\tau\xi - \pi\xi - (2\beta-\pi)(2k+1)\xi} d\xi \right. \\
&\quad \left. + \frac{e^{-(2\beta-\pi)(2k+1)/2}}{2(\lambda - e^{-(2\beta-\pi)(2k+1)/2})^2} \int_{-\delta}^{\delta} (\xi + ih) m_1(\xi) e^{i\tau\xi - \pi\xi - (2\beta-\pi)(2k+1)\xi} d\xi \right) \\
&= \sum_{k \geq 1} \frac{g_1^{(k)}(\tau)}{\lambda - e^{-(2\beta-\pi)(2k+1)/2}} + \frac{g_2^{(k)}(\tau) e^{-(2\beta-\pi)(2k+1)/2}}{(2\lambda - e^{-(2\beta-\pi)(2k+1)/2})^2},
\end{aligned}$$

where $g_1^{(k)}, g_2^{(k)}$ are entire functions of exponential type with bounds uniform in k .

Thus the above series converges to a function $E_4(\tau, \lambda)$, entire in τ and holomorphic when $|\lambda| > e^{-3(\beta-\pi/2)}$.

In order to conclude the proof, we finally need to analyze the sum of the terms involving I through III^* ; that is

$$\begin{aligned}
&\sum_{j \geq 0} (I - II + III) \lambda^j + \sum_{j < 0} (I^* - II^* + III^*) \lambda^j \\
&= \sum_{j \geq 0} (I + III) \lambda^j + \sum_{j < 0} (I^* + III^*) \lambda^j - \left(\sum_{j \geq 0} II \lambda^j + \sum_{j < 0} II^* \lambda^j \right).
\end{aligned}$$

As in the case of $E^{(1)}$, we split the argument in two.

10.8. The Case of $\sum_{j \geq 0} (I + III) \lambda^j + \sum_{j < 0} (I^* + III^*) \lambda^j$. Arguing as in (51) and (52), expanding m and \tilde{m} in Taylor series, we see that the above sum equals

$$\sum_{k, \ell=1}^{+\infty} \sum_{j \in \mathbb{Z}} \mathcal{I}_j^{(k, \ell)} \lambda^j,$$

where, for $j \geq 0$,

$$\begin{aligned}
\mathcal{I}_j^{(k, \ell)} &= \left(\int_{-\infty}^{-\delta} + \int_{\frac{j+1}{2} + \delta}^{+\infty} \right) \sigma_{(2\beta-\pi)(2\ell+1), (2k+1)\pi}(\xi) \\
&\quad \times (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) e^{i\xi\tau} e^{-(2k+1)\pi|\xi|} e^{-(2\beta-\pi)|\xi - \frac{j+1}{2}|} d\xi,
\end{aligned}$$

while for $j < 0$

$$\begin{aligned}
\mathcal{I}_j^{(k, \ell)} &= \left(\int_{-\infty}^{\frac{j+1}{2} - \delta} + \int_{\delta}^{+\infty} \right) \sigma_{(2\beta-\pi)(2\ell+1), (2k+1)\pi}(\xi) \\
&\quad \times (\xi + ih) \left(\xi + ih - \frac{j+1}{2} \right) e^{i\xi\tau} e^{-(2k+1)\pi|\xi|} e^{-(2\beta-\pi)|\xi - \frac{j+1}{2}|} d\xi.
\end{aligned}$$

By applying Lemmas 9.2 and 9.4 we see that the series above equals

$$\begin{aligned} \sum_{k,\ell=1}^{+\infty} \frac{e^{ih(R+S)-\delta[i\tau+R+S]}}{(i\tau+R+S)^2} & \left[\frac{e^{R/2}}{2(e^{R/2}-\lambda)^2} (1+\psi_1^+(\tau,\lambda)) \right. \\ & + \frac{e^{-[i\tau+S]/2}}{2(\lambda-e^{-[i\tau+S]/2})^2} (1+\psi_2^-(\tau,\lambda)) + \frac{e^{-[i\tau+S]/2}\psi_4^+(\tau)}{2(e^{R/2}-\lambda)(e^{-[i\tau+S]/2}-\lambda)} \\ & + \frac{e^{-ih(R+S)-\delta[R+S-i\tau]}}{(i\tau-R-S)^2} \left[\frac{e^{[S-i\tau]/2}}{2(e^{[S-i\tau]/2}-\lambda)^2} (1+\psi_2^+(\tau,\lambda)) \right. \\ & \left. \left. + \frac{e^{-R/2}}{2(\lambda-e^{-R/2})^2} (1+\psi_1^-(\tau,\lambda)) + \frac{e^{[S-i\tau]/2}\psi_4^-(\tau)}{2(e^{-R/2}-\lambda)(e^{[S-i\tau]/2}-\lambda)} \right] \right], \end{aligned}$$

where $R = (2\ell+1)(2\beta-\pi)$ and $S = (2k+1)\pi$; see formula (69).

We are going to show that the functions depending on k , that is on S , can be summed, and their sums are functions of (τ, λ) , holomorphic in a neighborhood of $\overline{\mathcal{D}}$, bounded (together with their derivatives) as $|\operatorname{Re} \tau| \rightarrow +\infty$.

We let (τ, λ) vary in the closure of the domain

$$\mathcal{D}_{4(\beta-2\pi), 2\pi} = \{ |\operatorname{Im} \tau - \log |\lambda|^2| < 2\pi, \ e^{-2(\beta-2\pi)} < |\lambda| < e^{2(\beta-2\pi)} \},$$

a domain that contains the closure of the domain \mathcal{D} . Using Lemma 9.2, we now obtain that

$$\begin{aligned} \left| \frac{e^{ih(R+S)-\delta[i\tau+R+S]}}{(i\tau+R+S)^2} \right| \cdot |e^{R/2}(1+\psi_1^+(\tau,\lambda))| & \leq C(|\operatorname{Re} \tau| + R + S) \frac{e^{-\delta[i\tau+R+S]}}{|i\tau+R+S|^2} \\ & \leq Ce^{-\delta(R+S)}, \end{aligned} \quad (62)$$

as $R, S \rightarrow +\infty$, uniformly in $(\tau, \lambda) \in \mathcal{D}_{2(\beta-2\pi), 2\pi}$.

Next notice that the functions $\frac{e^{-[i\tau+S]/2}}{2(\lambda-e^{-[i\tau+S]/2})^2}$ are bounded for $(\tau, \lambda) \in \mathcal{D}_{4(\beta-2\pi), 2\pi}$, uniformly in $S = (2k+1)\pi$ and $R = (2\ell+1)\pi$. Then

$$\begin{aligned} \left| \frac{e^{ih(R+S)-\delta[i\tau+R+S]}}{(i\tau+R+S)^2} \cdot \frac{e^{-[i\tau+S]/2}}{2(\lambda-e^{-[i\tau+S]/2})^2} (1+\psi_2^-(\tau,\lambda)) \right| \\ \leq Ce^{-\delta(R+S)} \frac{|\operatorname{Re} \tau| + R + S}{|i\tau+R+S|^2} \leq Ce^{-\delta(R+S)} \end{aligned} \quad (63)$$

as $R, S \rightarrow +\infty$, uniformly in $(\tau, \lambda) \in \mathcal{D}_{4(\beta-2\pi), 2\pi}$.

The arguments for the remaining terms are analogous to the ones in (56)–(59), so we obtain that they are all bounded by a constant times $e^{-\delta(R+S)}$. This completes the proof of this case.

Finally, we have

10.9. **The Case of $\sum_{j \geq 0} II \lambda^j + \sum_{j < 0} II^* \lambda^j$.** As in the previous Subsection 10.5 we use Lemmas 9.3 and 9.5, and summation by parts. We begin with the sum for $j \geq 0$. Since $\delta \leq \xi \leq \frac{j+1}{2} - \delta$, we can write

$$\begin{aligned}
& \sum_{j \geq 0} II \lambda^j \\
&= e^{ih(2\beta-2\pi)} \sum_{j \geq 0} e^{-(2\beta-\pi)\frac{j+1}{2}} \lambda^j \int_{\delta}^{\frac{j+1}{2}-\delta} [\xi^2 + b\xi + c] e^{(i\tau+(2\beta-\pi)-\pi)\xi} m_1(\xi) \tilde{m}(\xi) d\xi \\
&= e^{ih(2\beta-2\pi)} \sum_{j \geq 0} e^{-(2\beta-\pi)\frac{j+1}{2}} \lambda^j \left(\sum_{k=1}^N e^{-2\pi i k h} \int_{\delta}^{\frac{j+1}{2}-\delta} [\xi^2 + b\xi + c] e^{(i\tau+(2\beta-\pi)-(2k+1)\pi)\xi} d\xi \right. \\
&\quad \left. + \int_{\delta}^{\frac{j+1}{2}-\delta} [\xi^2 + b\xi + c] e^{(i\tau+(2\beta-\pi)-(2N+1)\pi)\xi} m_1(\xi) d\xi \right) \\
&\equiv A + B .
\end{aligned}$$

Now we only sketch the details. In order to evaluate A we apply Lemma 9.3 with $R = 2\beta - \pi$ and $S = (2k+1)\pi$, $k = 1, \dots, N$. In order to evaluate B we use summation by parts (see 60) and argue as in Subsection 10.5 This concludes the proof of Proposition 7.3. \square

11. COMPLETION OF THE PROOF OF PROPOSITION 9.1.

In this final section we complete the proof Proposition 9.1.

End of the proof of Proposition 9.1. We are in a position now to conclude the proof of Proposition 9.1, modulo proving the lemmas from Section 9. We shall also prove those lemmas at this time (after the proof of the proposition). Using Lemmas 9.2 through 9.5 we obtain that

$$\begin{aligned}
& \sum_{j \geq 0} (I - II + III) \lambda^j + \sum_{j \leq -1} (I^* - II^* + III^*) \lambda^j \\
&= \frac{e^{ih(R+S)+R/2}}{2(e^{R/2} - \lambda)^2(i\tau + R + S)^2} e^{-\delta[i\tau+R+S]} (1 + \psi_1^+(\tau, \lambda)) \\
&\quad + \frac{e^{-ih(R+S)+[S-i\tau]/2}}{2(e^{[S-i\tau]/2} - \lambda)^2(i\tau - R - S)^2} e^{-\delta[R+S-i\tau]} (1 + \psi_2^+(\tau, \lambda)) \\
&\quad + \frac{e^{ih(R-S)-\delta[S-R-i\tau]+R/2}}{2(e^{[S-i\tau]/2} - \lambda)^2(e^{R/2} - \lambda)^2} \psi_3^+(\tau, \lambda)
\end{aligned} \tag{64}$$

$$\begin{aligned}
& + \left(\frac{2e^{ih(R+S)-\delta[i\tau+R+S]}}{(e^{R/2}-\lambda)(i\tau+R+S)^3} - \frac{2e^{-ih(R+S)-\delta[R+S-i\tau]}}{(e^{[S-i\tau]/2}-\lambda)(i\tau-R-S)^3} \right) \\
& + \frac{e^{-ih(R+S)-R/2}}{2(\lambda-e^{-R/2})^2(i\tau-R-S)^2} e^{-\delta[R+S-i\tau]} (1 + \psi_1^-(\tau, \lambda)) \\
& + \frac{e^{ih(R+S)} e^{-[i\tau+S]/2}}{2(\lambda-e^{-[i\tau+S]/2})^2(i\tau+R+S)^2} e^{-\delta[i\tau+R+S]} (1 + \psi_2^-(\tau, \lambda)) \\
& + \frac{e^{-ih(R-S)+\delta[i\tau-R+S]-R/2}}{2(e^{-[i\tau+S]/2}-\lambda)^2(e^{-R/2}-\lambda)^2} \psi_3^-(\tau, \lambda) \\
& + \left(\frac{2e^{-ih(R+S)-\delta[R+S-i\tau]}}{(e^{-R/2}-\lambda)(i\tau-R-S)^3} - \frac{2e^{ih(R+S)-\delta[i\tau+R+S]}}{(e^{-[i\tau+S]/2}-\lambda)(i\tau+R+S)^3} \right). \quad (65)
\end{aligned}$$

Previously we have alluded to a certain cancellation of the cubic terms. We expect this because the worm domain is locally like a product and the kernels for such domains have degree-two singularities (see [APF]). Thus no terms with third-order singularity should be present. Now we concentrate on the cubic terms and calculate to confirm this expectation.

Notice that

$$\begin{aligned}
& \frac{1}{(e^{-R/2}-\lambda)(i\tau-R-S)^3} - \frac{1}{(e^{[S-i\tau]/2}-\lambda)(i\tau-R-S)^3} \\
& = \frac{1}{(i\tau-R-S)^3} \frac{e^{[S-i\tau]/2} - e^{-R/2}}{(e^{-R/2}-\lambda)(e^{[S-i\tau]/2}-\lambda)} \\
& = \frac{e^{[S-i\tau]/2} \psi_4^-(\tau)}{2(i\tau-R-S)^2(e^{-R/2}-\lambda)(e^{[S-i\tau]/2}-\lambda)}. \quad (66)
\end{aligned}$$

Analogously,

$$\begin{aligned}
& \frac{1}{(e^{R/2}-\lambda)(i\tau+R+S)^3} - \frac{1}{(e^{-[i\tau+S]/2}-\lambda)(i\tau+R+S)^3} \\
& = \frac{e^{-[i\tau+S]/2} \psi_4^+(\tau)}{2(i\tau+R+S)^2(e^{R/2}-\lambda)(e^{-[i\tau+S]/2}-\lambda)}, \quad (67)
\end{aligned}$$

where, in the last two displays,

$$\psi_4^\pm(\tau) = E(i\tau \pm (R+S)), \quad \text{with} \quad E(x) = \frac{1 - e^{x/2}}{x/2}. \quad (68)$$

Now we plug formulas (66), (67) into (65) and factor common powers once again. In total, we finally obtain that the sum on the lefthand side of (65) equals

$$\begin{aligned}
& \frac{e^{ih(R+S)-\delta[i\tau+R+S]}}{(i\tau+R+S)^2} \left[\frac{e^{R/2}}{2(e^{R/2}-\lambda)^2} (1+\psi_1^+(\tau, \lambda)) \right. \\
& \quad \left. + \frac{e^{-[i\tau+S]/2}}{2(\lambda-e^{-[i\tau+S]/2})^2} (1+\psi_2^-(\tau, \lambda)) + \frac{e^{-[i\tau+S]/2}\psi_4^+(\tau)}{2(e^{R/2}-\lambda)(e^{-[i\tau+S]/2}-\lambda)} \right] \\
& + \frac{e^{-ih(R+S)-\delta[R+S-i\tau]}}{(i\tau-R-S)^2} \left[\frac{e^{[S-i\tau]/2}}{2(e^{[S-i\tau]/2}-\lambda)^2} (1+\psi_2^+(\tau, \lambda)) \right. \\
& \quad \left. + \frac{e^{-R/2}}{2(\lambda-e^{-R/2})^2} (1+\psi_1^-(\tau, \lambda)) + \frac{e^{[S-i\tau]/2}\psi_4^-(\tau)}{2(e^{-R/2}-\lambda)(e^{[S-i\tau]/2}-\lambda)} \right] \\
& + \frac{e^{ih(R-S)-\delta[S-R-i\tau]+R/2}}{2(e^{[S-i\tau]/2}-\lambda)^2(e^{R/2}-\lambda)^2} \psi_3^+(\tau, \lambda) \\
& + \frac{e^{-ih(R-S)+\delta[i\tau-R+S]-R/2}}{2(e^{-[i\tau+S]/2}-\lambda)^2(e^{-R/2}-\lambda)^2} \psi_3^-(\tau, \lambda) . \tag{69}
\end{aligned}$$

If we set $\delta = 0$ now, we obtain the expression of the sum as in the statement of Proposition 9.1.

Finally, notice that from (68) it follows at once that

$$|\psi_4^\pm(\tau, \lambda)| \leq C_M e^{\pm(R+S)/2} . \tag{70}$$

This proves Proposition 9.1. \square

Finally, we need to prove Lemmas 9.2–9.5.

Proof of Lemma 9.2. We have

$$I = e^{ih(R+S)} e^{-R\frac{j+1}{2}} e^{-\delta[i\tau+R+S]} \left\{ \frac{c - \delta b + \delta^2}{i\tau + R + S} - \frac{b - 2\delta}{(i\tau + R + S)^2} + \frac{2}{(i\tau + R + S)^3} \right\}$$

and

$$\begin{aligned}
III = & -e^{-ih(R+S)-\delta[R+S-i\tau]} e^{\frac{j+1}{2}[i\tau-S]} \left\{ \left(\frac{j+1}{2} + \delta \right)^2 \frac{1}{i\tau - R - S} \right. \\
& + \left(\frac{j+1}{2} + \delta \right) \left(\frac{b}{i\tau - R - S} - \frac{2}{(i\tau - R - S)^2} \right) \\
& \left. + \left(\frac{c}{i\tau - R - S} - \frac{b}{(i\tau - R - S)^2} + \frac{2}{(i\tau - R - S)^3} \right) \right\} .
\end{aligned}$$

Therefore, recalling that $b = 2ih - (j+1)/2$ and $c = -h^2 - ih(j+1)/2$, we see that

$I + III$

$$\begin{aligned}
&= e^{ih(R+S)-\delta[i\tau+R+S]} e^{-R\frac{i+1}{2}} \left\{ \frac{j+1}{2} \left(\frac{1}{(i\tau+R+S)^2} + \frac{\delta-ih}{i\tau+R+S} \right) \right. \\
&\quad \left. + \left(\frac{2}{(i\tau+R+S)^3} + \frac{2(\delta-ih)}{(i\tau+R+S)^2} + \frac{(\delta-ih)^2}{i\tau+R+S} \right) \right\} \\
&\quad - e^{\frac{i+1}{2}[i\tau-S]} \left\{ e^{-ih(R+S)-\delta[R+S-i\tau]} \frac{j+1}{2} \left(\frac{\delta+ih}{i\tau-R-S} - \frac{1}{(i\tau-R-S)^2} \right) \right. \\
&\quad \left. + e^{-ih(R+S)-\delta[R+S-i\tau]} \left(\frac{(\delta+ih)^2}{i\tau-R-S} - \frac{2(\delta+ih)}{(i\tau-R-S)^2} + \frac{2}{(i\tau-R-S)^3} \right) \right\}.
\end{aligned}$$

Applying the summation formulas (46) to our last expression we obtain for

$$|e^{-R/2}\lambda| < 1 \quad \text{and} \quad |e^{[i\tau-S]/2}\lambda| < 1,$$

that is for $(\tau, \lambda) \in \mathcal{D}_{R,S}$, that

$$\begin{aligned}
&\sum_{j \geq 0} (I + III) \lambda^j \\
&= \frac{e^{R/2}}{2(e^{R/2} - \lambda)^2} \left[e^{ih(R+S)-\delta[i\tau+R+S]} \left(\frac{1}{(i\tau+R+S)^2} + \frac{\delta-ih}{i\tau+R+S} \right) \right] \\
&\quad + \frac{1}{e^{R/2} - \lambda} \left[e^{ih(R+S)-\delta[i\tau+R+S]} \left(\frac{2}{(i\tau+R+S)^3} + \frac{2(\delta-ih)}{(i\tau+R+S)^2} + \frac{(\delta-ih)^2}{i\tau+R+S} \right) \right] \\
&\quad + \frac{e^{[S-i\tau]/2}}{2(e^{[S-i\tau]/2} - \lambda)^2} \left[e^{-ih(R+S)-\delta[R+S-i\tau]} \left(\frac{1}{(i\tau-R-S)^2} - \frac{\delta+ih}{i\tau-R-S} \right) \right] \\
&\quad - \frac{1}{e^{[S-i\tau]/2} - \lambda} \left[e^{-ih(R+S)-\delta[R+S-i\tau]} \left(\frac{2}{(i\tau-R-S)^3} - \frac{2(\delta+ih)}{(i\tau-R-S)^2} + \frac{(\delta+ih)^2}{i\tau-R-S} \right) \right].
\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{j \geq 0} (I + III) \lambda^j \\
&= \frac{e^{ih(R+S)-\delta[i\tau+R+S]+R/2}}{2(e^{R/2} - \lambda)^2} \left[\frac{1}{(i\tau+R+S)^2} + \frac{\delta-ih}{i\tau+R+S} \right] \\
&\quad + \frac{e^{ih(R+S)-\delta[i\tau+R+S]}}{e^{R/2} - \lambda} \left[\frac{2}{(i\tau+R+S)^3} + \frac{2(\delta-ih)}{(i\tau+R+S)^2} + \frac{(\delta-ih)^2}{i\tau+R+S} \right] \\
&\quad + \frac{e^{-ih(R+S)-\delta[R+S-i\tau]+[S-i\tau]/2}}{2(e^{[S-i\tau]/2} - \lambda)^2} \left[\frac{1}{(i\tau-R-S)^2} - \frac{\delta+ih}{i\tau-R-S} \right] \\
&\quad + \frac{e^{-ih(R+S)-\delta[R+S-i\tau]}}{e^{[S-i\tau]/2} - \lambda} \left[-\frac{2}{(i\tau-R-S)^3} + \frac{2(\delta+ih)}{(i\tau-R-S)^2} - \frac{(\delta+ih)^2}{i\tau-R-S} \right].
\end{aligned}$$

(Later on we shall see that the terms here that are of cubic order will cancel with the cubic terms that come from summing j with $j < 0$.)

Factoring out common powers, we may simplify our expression further and finally write

$$\begin{aligned}
& \sum_{j \geq 0} (I + III) \lambda^j \\
&= \frac{e^{ih(R+S)+R/2}}{2(e^{R/2} - \lambda)^2(i\tau + R + S)^2} e^{-\delta[i\tau + R + S]} (1 + \psi_1^+(\tau, \lambda)) \\
&+ \frac{e^{-ih(R+S)+[S-i\tau]/2}}{2(e^{[S-i\tau]/2} - \lambda)^2(i\tau - R - S)^2} e^{-\delta[R + S - i\tau]} (1 + \psi_2^+(\tau, \lambda)) \\
&+ \left(\frac{2e^{ih(R+S)-\delta[i\tau + R + S]}}{(e^{R/2} - \lambda)(i\tau + R + S)^3} - \frac{2e^{-ih(R+S)-\delta[R + S - i\tau]}}{(e^{[S-i\tau]/2} - \lambda)(i\tau - R - S)^3} \right), \quad (71)
\end{aligned}$$

where

$$\begin{aligned}
\psi_1^+(\tau, \lambda) &= (\delta - ih)(i\tau + R + S) \\
&\quad + 2(1 - e^{-R/2}\lambda) [2(\delta - ih) + (\delta - ih)^2(i\tau + R + S)], \\
\psi_2^+(\tau, \lambda) &= -(\delta + ih)(i\tau - R - S) \\
&\quad + 2(1 - e^{-[S-i\tau]/2}\lambda) [2(\delta + ih) - (\delta + ih)^2(i\tau - R - S)]. \quad (72)
\end{aligned}$$

Finally, the estimate for ψ_1^+ and ψ_2^+ follows at once from the explicit expressions (72). \square

In order to calculate the sum for II (Lemma 9.3), we need the following elementary but somewhat tedious lemma, whose proof is, once again, postponed.

Lemma 11.1. *Let $\alpha > 0$ and $\delta \geq 0$ be fixed. We let $f(x) = (\alpha e^{-x/2} - \lambda)^{-1}$, $\Psi_{\pm}(x) = e^{\mp 2\delta x} f(x)$, and define*

$$\begin{aligned}
Q_{\pm\delta}(x) &= \frac{ihx - 1}{x^2} \left[2 \frac{\Psi_{\pm}(x) - \Psi_{\pm}(0)}{x} - (\Psi'_{\pm}(x) + \Psi'_{\pm}(0)) \right] \\
&\quad + \frac{1}{x} \left[(h^2 - \delta^2)(\Psi_{\pm}(x) - \Psi_{\pm}(0)) \mp \delta(e^{\mp 2\delta x} f'(x) - f'(0)) \right]. \quad (73)
\end{aligned}$$

Then there exist entire functions $G_1^{\pm}, \dots, G_6^{\pm}$, independent of α and λ , such that

$$\begin{aligned}
Q_{\pm\delta}(x) &= \frac{\alpha}{2(\alpha e^{-x/2} - \lambda)^2(\alpha - \lambda)^2} \left[(\alpha e^{-x/2} - \lambda)(\alpha - \lambda)G_1^{\pm}(x) + (\alpha - \lambda)^2 G_2^{\pm}(x) \right. \\
&\quad \left. + \frac{(\alpha - \lambda)(\alpha e^{-x/2} - \lambda)^2}{\alpha} G_3^{\pm}(x) + \alpha \lambda G_4^{\pm}(x) + \lambda G_5^{\pm}(x) + \alpha^2 G_6^{\pm}(x) \right]. \quad (74)
\end{aligned}$$

The proof of Lemma 11.1 occurs at the very end of this paper.

Proof of Lemma 9.3. Using (42) we have that

$$\begin{aligned} II = e^{ih(R-S)} e^{-R\frac{j+1}{2}} & \left\{ \left[\left(\frac{j+1}{2} - \delta \right)^2 \frac{1}{i\tau + R - S} \right. \right. \\ & + \left(\frac{j+1}{2} - \delta \right) \left(\frac{b}{i\tau + R - S} - \frac{2}{(i\tau + R - S)^2} \right) \\ & + \left. \left(\frac{c}{i\tau + R - S} - \frac{b}{(i\tau + R - S)^2} + \frac{2}{(i\tau + R - S)^3} \right) \right] e^{(\frac{j+1}{2} - \delta)[i\tau + R - S]} \\ & - \left. \left[\frac{c + \delta b + \delta^2}{i\tau + R - S} - \frac{b}{(i\tau + R - S)^2} + \frac{2}{(i\tau + R - S)^3} \right] e^{\delta[i\tau + R - S]} \right\}. \end{aligned}$$

Therefore, recalling that $b = 2ih - (j+1)/2$ and $c = -h^2 - ih(j+1)/2$, we see that

$$\begin{aligned} II = e^{\frac{j+1}{2}[i\tau - S]} & \left\{ e^{ih(R-S) - \delta[i\tau + R - S]} \frac{j+1}{2} \left(\frac{ih - \delta}{i\tau + R - S} - \frac{1}{(i\tau + R - S)^2} \right) \right. \\ & + e^{ih(R-S) - \delta[i\tau + R - S]} \left(\frac{(\delta - ih)^2}{i\tau + R - S} + \frac{2(\delta - ih)}{(i\tau + R - S)^2} + \frac{2}{(i\tau + R - S)^3} \right) \\ & - e^{ih(R-S) + \delta[i\tau + R - S]} e^{-R\frac{j+1}{2}} \left\{ \frac{j+1}{2} \left(\frac{1}{(i\tau + R - S)^2} - \frac{\delta + ih}{i\tau + R - S} \right) \right. \\ & + \left. \left(\frac{2}{(i\tau + R - S)^3} - \frac{2(\delta + ih)}{(i\tau + R - S)^2} + \frac{(\delta + ih)^2}{i\tau + R - S} \right) \right\} \left. \right\}. \end{aligned}$$

Applying the summation formulas (46) we obtain that, for

$$|e^{-R/2}\lambda| < 1 \quad \text{and} \quad |e^{[i\tau - S]/2}\lambda| < 1,$$

that is for $(\tau, \lambda) \in \mathcal{D}_{R,S}$,

$$\begin{aligned} & \sum_{j \geq 0} II \lambda^j \\ &= \frac{e^{R/2}}{2(e^{R/2} - \lambda)^2} e^{ih(R-S) + \delta[i\tau + R - S]} \left(-\frac{\delta + ih}{i\tau + R - S} + \frac{1}{(i\tau + R - S)^2} \right) \\ &+ \frac{1}{e^{R/2} - \lambda} \left[e^{ih(R-S) + \delta[i\tau + R - S]} \left(\frac{2}{(i\tau + R - S)^3} - \frac{2(\delta + ih)}{(i\tau + R - S)^2} + \frac{(\delta + ih)^2}{i\tau + R - S} \right) \right] \\ &+ \frac{e^{[S - i\tau]/2}}{2(e^{[S - i\tau]/2} - \lambda)^2} \left[e^{ih(R-S) - \delta[i\tau + R - S]} \left(\frac{1}{(i\tau + R - S)^2} + \frac{(\delta + ih)}{i\tau + R - S} \right) \right] \\ &- \frac{1}{e^{[S - i\tau]/2} - \lambda} \left[e^{ih(R-S) - \delta[i\tau + R - S]} \left(\frac{2}{(i\tau + R - S)^3} + \frac{2(\delta - ih)}{(i\tau + R - S)^2} + \frac{(\delta - ih)^2}{i\tau + R - S} \right) \right]. \end{aligned} \tag{75}$$

Notice that there is an apparent singularity as $i\tau \rightarrow -(R - S)$. We now analyze the terms containing negative powers of $(i\tau + R - S)$ and show that they actually give rise to some cancellation. But we shall also see that there genuinely *are* singularities as $i\tau \rightarrow -(R + S)$ and $i\tau \rightarrow (R + S)$.

Temporarily, we introduce the notation

$$x = i\tau + R - S, \quad \text{and} \quad \alpha = e^{R/2}$$

so that

$$S - i\tau = R - x \quad \text{and} \quad e^{[S-i\tau]/2} = \alpha e^{-x/2}.$$

We now show that the ostensible singularities in $x = 0$ in fact all cancel out. Then we may rewrite the terms containing negative powers of $(i\tau + R - S)$ (that is, of x) on the righthand side of formula (75) as

$$\begin{aligned} e^{ih(R-S)} & \left\{ \frac{1}{x^3} \left[\frac{2e^{\delta x}}{\alpha - \lambda} - \frac{2e^{-\delta x}}{\alpha e^{-x/2} - \lambda} \right] \right. \\ & + \frac{1}{x^2} \left[\frac{\alpha e^{-x/2} e^{-\delta x}}{2(\alpha e^{-x/2} - \lambda)^2} + \frac{2(ih - \delta)e^{-\delta x}}{\alpha e^{-x/2} - \lambda} + \frac{\alpha e^{\delta x}}{2(\alpha - \lambda)^2} - \frac{2(\delta + ih)e^{\delta x}}{\alpha - \lambda} \right] \\ & \left. + \frac{1}{x} \left[\frac{(\delta - ih)\alpha e^{-x/2} e^{-\delta x}}{2(\alpha e^{-x/2} - \lambda)^2} - \frac{(\delta - ih)^2 e^{-\delta x}}{\alpha e^{-x/2} - \lambda} - \frac{(\delta + ih)\alpha e^{\delta x}}{2(\alpha - \lambda)^2} + \frac{(\delta + ih)^2 e^{\delta x}}{\alpha - \lambda} \right] \right\}. \end{aligned} \quad (76)$$

Now if we set

$$f(x) = \frac{1}{\alpha e^{-x/2} - \lambda}, \quad \text{and} \quad F(x) = e^{-2\delta x} f(x),$$

then we see that we may rewrite the above expression (76) as $e^{ih(R-S)+\delta x} Q_\delta(x)$, where Q_δ is given by (73).

Next we use Lemma 11.1 to see that Q_δ is regular in $x = 0$ and to write

$$\begin{aligned} Q_\delta(x) &= \frac{\alpha}{2(\alpha e^{-x/2} - \lambda)^2(\alpha - \lambda)^2} \left[(\alpha e^{-x/2} - \lambda)(\alpha - \lambda)G_1^+(x) + (\alpha - \lambda)^2 G_2^+(x) \right. \\ & \quad \left. + \frac{(\alpha - \lambda)(\alpha e^{-x/2} - \lambda)^2}{\alpha} G_3^+(x) + \alpha \lambda G_4^+(x) + \lambda G_5^+(x) + \alpha^2 G_6^+(x) \right], \end{aligned}$$

where G_1^+, \dots, G_6^+ are the functions appearing in Lemma 11.1 and whose explicit expression is given in (85).

Therefore the above expression (76) equals

$$e^{ih(R-S)+\delta[i\tau+R-S]} Q_\delta(i\tau + R - S) = \frac{e^{ih(R-S)+\delta[i\tau+R-S]+R/2}}{2(e^{[S-i\tau]/2} - \lambda)^2(e^{R/2} - \lambda)^2} \psi_3^+(\tau, \lambda),$$

where we have set

$$\varphi_j^+(\tau) = G_j^+(i\tau + R - S), \quad \text{for } j = 1, \dots, 6, \quad (77)$$

and

$$\begin{aligned} \psi_3^+(\tau, \lambda) &= (e^{[S-i\tau]/2} - \lambda)(e^{R/2} - \lambda)\varphi_1^+(\tau) + (e^{R/2} - \lambda)^2\varphi_2^+(\tau) \\ &\quad + (1 - e^{-R/2}\lambda)(e^{[S-i\tau]/2} - \lambda)^2\varphi_3^+(\tau) + e^{R/2}\lambda\varphi_4^+(\tau) + \lambda\varphi_5^+(\tau) + e^R\varphi_6^+(\tau). \end{aligned} \quad (78)$$

In view of the cancellation of negative powers of x —in other words, any dependency on x is in fact bounded and smooth—we may simplify formula (75) to obtain

$$\sum_{j \geq 0} II\lambda^j = \frac{e^{ih(R-S) - \delta[S-R-i\tau] + R/2}}{2(e^{[S-i\tau]/2} - \lambda)^2(e^{R/2} - \lambda)^2} \psi_3^+(\tau, \lambda). \quad (79)$$

Recalling definition (78)), we write ψ_3^+ explicitly as

$$\begin{aligned} \psi_3^+(\tau, \lambda) &= (e^{[S-i\tau]/2} - \lambda)(e^{R/2} - \lambda)G_1^+(i\tau + R - S) + (e^{R/2} - \lambda)^2G_2^+(i\tau + R - S) \\ &\quad + (1 - e^{-R/2}\lambda)^2(e^{[S-i\tau]/2} - \lambda)G_3^+(i\tau + R - S) + e^{R/2}\lambda G_4^+(i\tau + R - S) \\ &\quad + \lambda(\alpha - \lambda)G_5^+(i\tau + R - S) + e^R G_6^+(i\tau + R - S), \end{aligned}$$

where G_1^+, \dots, G_6^+ are given in (85). Then, since these are entire functions, we have

$$\begin{aligned} |\psi_3^+(\tau, \lambda)| &\leq C_M \left(e^{(R+S)/2} |G_1^+(i\tau + R - S)| + e^{R/2} |G_2^+(i\tau + R - S)| \right. \\ &\quad \left. + e^{S/2} |G_3^+(i\tau + R - S)| + e^{R/2} (|G_4^+(i\tau + R - S)| + |G_5^+(i\tau + R - S)|) \right. \\ &\quad \left. + e^R |G_6^+(i\tau + R - S)| \right) \\ &\leq C_M (e^{R+S/2} + e^{S(2\delta+1/2)} + e^{2R}). \end{aligned}$$

This proves Lemma 9.3. \square

Proof of Lemma 9.4. Recall formula (42) and that $b = 2ih - (j+1)/2$ and $c = -h^2 - ih(j+1)/2$. We calculate:

$$\begin{aligned} I^* &= e^{ih(R+S)} e^{-R\frac{j+1}{2}} \left\{ \left[\frac{(\frac{j+1}{2} - \delta)^2}{i\tau + R + S} + \left(\frac{2ih - \frac{j+1}{2}}{i\tau + R + S} - \frac{2}{(i\tau + R + S)^2} \right) \left(\frac{j+1}{2} - \delta \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{2}{(i\tau + R + S)^3} - \frac{2ih - \frac{j+1}{2}}{(i\tau + R + S)^2} + \frac{ih(ih - \frac{j+1}{2})}{i\tau + R + S} \right) \right] e^{(\frac{j+1}{2} - \delta)(i\tau + R + S)} \right\} \\ &= e^{ih(R+S) - \delta[i\tau + R + S]} e^{[i\tau + S]\frac{j+1}{2}} \left\{ \frac{2}{(i\tau + R + S)^3} + \frac{2(\delta - ih)}{(i\tau + R + S)^2} + \frac{(\delta - ih)^2}{(i\tau + R + S)} \right. \\ &\quad \left. - \frac{j+1}{2} \left(\frac{1}{(i\tau + R + S)^2} + \frac{\delta - ih}{i\tau + R + S} \right) \right\}. \end{aligned}$$

Next, for the term III^* , we have

$$\begin{aligned}
III^* &= -e^{-ih(R+S)-\delta[R+S-i\tau]} e^{R\frac{i+1}{2}} \left\{ \left(\frac{2}{(i\tau - R - S)^3} - \frac{2ih - \frac{i+1}{2} + 2\delta}{(i\tau - R - S)^2} \right. \right. \\
&\quad \left. \left. + \frac{ih(ih - \frac{i+1}{2}) + (2ih - \frac{i+1}{2})\delta + \delta^2}{i\tau - R - S} \right) \right\} \\
&= -e^{-ih(R+S)-\delta[R+S-i\tau]} e^{R\frac{i+1}{2}} \left\{ \frac{2}{(i\tau - R - S)^3} - \frac{2(\delta + ih)}{(i\tau - R - S)^2} + \frac{(\delta + ih)^2}{i\tau - R - S} \right. \\
&\quad \left. + \frac{j+1}{2} \left(\frac{1}{(i\tau - R - S)^2} - \frac{\delta + ih}{i\tau - R - S} \right) \right\}.
\end{aligned}$$

Now we must sum in j . Since

$$\sum_{j \leq -1} e^{\gamma \frac{j+1}{2}} \lambda^j = \frac{1}{\lambda - e^{-\gamma/2}} \quad \text{and} \quad \sum_{j \leq -1} \frac{j+1}{2} e^{\gamma \frac{j+1}{2}} \lambda^j = -\frac{e^{-\gamma/2}}{2(\lambda - e^{-\gamma/2})^2},$$

when $|e^{\gamma/2}\lambda| > 1$, i.e. when $|\lambda| > e^{-R/2}$, we have that

$$\begin{aligned}
&\sum_{j \leq -1} (I^* + III^*) \lambda^j \\
&= \frac{1}{\lambda - e^{-R/2}} \left[e^{-ih(R+S)-\delta[R+S-i\tau]} \left(-\frac{2}{(i\tau - R - S)^3} + \frac{2(\delta + ih)}{(i\tau - R - S)^2} - \frac{(\delta + ih)^2}{i\tau - R - S} \right) \right] \\
&\quad + \frac{e^{-R/2}}{2(\lambda - e^{-R/2})^2} \left[e^{-ih(R+S)-\delta[R+S-i\tau]} \left(\frac{1}{(i\tau - R - S)^2} + \frac{\delta + ih}{i\tau - R - S} \right) \right] \\
&\quad + \frac{1}{\lambda - e^{-[i\tau+S]/2}} \left[e^{ih(R+S)-\delta[i\tau+R+S]} \left(\frac{2}{(i\tau + R + S)^3} + \frac{2(\delta - ih)}{(i\tau + R + S)^2} + \frac{(\delta - ih)^2}{i\tau + R + S} \right) \right] \\
&\quad + \frac{e^{-[i\tau+S]/2}}{2(\lambda - e^{-[i\tau+S]/2})^2} \left[e^{ih(R+S)-\delta[i\tau+R+S]} \left(\frac{1}{(i\tau + R + S)^2} + \frac{\delta - ih}{i\tau + R + S} \right) \right] \Bigg\}.
\end{aligned}$$

Therefore, factoring common denominators, we we may simplify the above expression and obtain that

$$\begin{aligned}
& \sum_{j \leq -1} (I^* + III^*) \lambda^j \\
&= e^{-ih(R+S)-\delta[R+S-i\tau]} \left\{ \frac{1}{\lambda - e^{-R/2}} \left[-\frac{2}{(i\tau - R - S)^3} + \frac{2(\delta + ih)}{(i\tau - R - S)^2} - \frac{(\delta + ih)^2}{i\tau - R - S} \right] \right. \\
&\quad \left. + \frac{e^{-R/2}}{2(\lambda - e^{-R/2})^2} \left[\frac{1}{(i\tau - R - S)^2} - \frac{\delta + ih}{i\tau - R - S} \right] \right\} \\
&\quad + e^{ih(R+S)-\delta[i\tau+R+S]} \left\{ \frac{1}{\lambda - e^{-[i\tau+S]/2}} \left[\frac{2}{(i\tau + R + S)^3} + \frac{2(\delta - ih)}{(i\tau + R + S)^2} + \frac{(\delta - ih)^2}{i\tau + R + S} \right] \right. \\
&\quad \left. + \frac{e^{-[i\tau+S]/2}}{2(\lambda - e^{-[i\tau+S]/2})^2} \left[\frac{1}{(i\tau + R + S)^2} + \frac{\delta - ih}{i\tau + R + S} \right] \right\} \\
&= \frac{e^{-ih(R+S)-R/2}}{2(\lambda - e^{-R/2})^2(i\tau - R - S)^2} e^{-\delta[R+S-i\tau]} (1 + \psi_1^-(\tau, \lambda)) \\
&\quad + \frac{e^{ih(R+S)} e^{-[i\tau+S]/2}}{2(\lambda - e^{-[i\tau+S]/2})^2(i\tau + R + S)^2} e^{-\delta[i\tau+R+S]} (1 + \psi_2^-(\tau, \lambda)) \\
&\quad + \left(\frac{2e^{-ih(R+S)-\delta[R+S-i\tau]}}{(e^{-R/2} - \lambda)(i\tau - R - S)^3} - \frac{2e^{ih(R+S)-\delta[i\tau+R+S]}}{(e^{-[i\tau+S]/2} - \lambda)(i\tau + R + S)^3} \right), \tag{80}
\end{aligned}$$

where

$$\begin{aligned}
\psi_1^-(\tau, \lambda) &= -(\delta + ih)(i\tau - R - S) \\
&\quad + 2(e^{R/2}\lambda - 1)[2(\delta + ih) - (\delta + ih)^2(i\tau - R - S)], \\
\psi_2^-(\tau, \lambda) &= (\delta - ih)(i\tau + R + S) \\
&\quad + 2(\lambda e^{[i\tau+S]/2} - 1)[2(\delta - ih) + (\delta - ih)^2(i\tau + R + S)]. \tag{81}
\end{aligned}$$

Finally, the estimate for ψ_1^- and ψ_2^- follows at once from the explicit expressions above. \square

Proof of Lemma 9.5. Recalling formula (42), for the term II^* we have

$$\begin{aligned}
II^* &= e^{ih(S-R)} e^{R\frac{j+1}{2}} \left\{ \left(\frac{2}{(i\tau - R + S)^3} + \frac{2(\delta - ih) + \frac{j+1}{2}}{(i\tau - R + S)^2} + \frac{(\delta - ih)^2 + (\delta - ih)\frac{j+1}{2}}{(i\tau - R + S)} \right) e^{-\delta[i\tau - R + S]} \right. \\
&\quad - \left[\frac{(\frac{j+1}{2} + \delta)^2}{(i\tau - R + S)} + \left(\frac{2ih - \frac{j+1}{2}}{i\tau - R + S} - \frac{2}{(i\tau - R + S)^2} \right) \left(\frac{j+1}{2} + \delta \right) \right. \\
&\quad \left. \left. + \left(\frac{2}{(i\tau - R + S)^3} - \frac{2ih - \frac{j+1}{2}}{(i\tau - R + S)^2} + \frac{ih(ih - \frac{j+1}{2})}{i\tau - R + S} \right) \right] e^{(\frac{j+1}{2} + \delta)[i\tau - R + S]} \right\} \\
&= e^{ih(S-R) - \delta[i\tau - R + S]} e^{R\frac{j+1}{2}} \left\{ \frac{2}{(i\tau - R + S)^3} + \frac{2(\delta - ih)}{(i\tau - R + S)^2} + \frac{(\delta - ih)^2}{i\tau - R + S} \right. \\
&\quad \left. + \frac{j+1}{2} \left(\frac{1}{(i\tau - R + S)^2} + \frac{\delta - ih}{i\tau - R + S} \right) \right\} \\
&\quad - e^{ih(S-R) + \delta[i\tau - R + S]} e^{[i\tau + S]\frac{j+1}{2}} \left\{ \frac{2}{(i\tau - R + S)^3} - \frac{2(\delta + ih)}{(i\tau - R + S)^2} + \frac{(\delta + ih)^2}{i\tau - R + S} \right. \\
&\quad \left. + \frac{j+1}{2} \left(-\frac{1}{(i\tau - R + S)^2} + \frac{(\delta + ih)}{i\tau - R + S} \right) \right\}.
\end{aligned}$$

Now we must sum in j . Since

$$\sum_{j \leq -1} e^{\gamma \frac{j+1}{2}} \lambda^j = \frac{1}{\lambda - e^{-\gamma/2}} \quad \text{and} \quad \sum_{j \leq -1} \frac{j+1}{2} e^{\gamma \frac{j+1}{2}} \lambda^j = -\frac{e^{-\gamma/2}}{2(\lambda - e^{-\gamma/2})^2},$$

when $|e^{\gamma/2} \lambda| > 1$, i.e. when $|\lambda| > e^{-R/2}$, we have that

$$\begin{aligned}
&\sum_{j \leq -1} II^* \lambda^j \\
&= \frac{1}{\lambda - e^{-R/2}} \left[e^{-ih(R-S) - \delta[i\tau - R + S]} \left(-\frac{2}{(i\tau - R + S)^3} - \frac{2(\delta - ih)}{(i\tau - R + S)^2} - \frac{(\delta - ih)^2}{i\tau - R + S} \right) \right. \\
&\quad + \frac{e^{-R/2}}{2(\lambda - e^{-R/2})^2} \left[e^{-ih(R-S) - \delta[i\tau - R + S]} \left(\frac{1}{(i\tau - R + S)^2} + \frac{\delta - ih}{i\tau - R + S} \right) \right. \\
&\quad + \frac{1}{\lambda - e^{-[i\tau + S]/2}} \left[e^{-ih(R-S) + \delta[i\tau - R + S]} \left(\frac{2}{(i\tau - R + S)^3} - \frac{2(\delta + ih)}{(i\tau - R + S)^2} + \frac{(\delta + ih)^2}{i\tau - R + S} \right) \right. \\
&\quad \left. \left. + \frac{e^{-[i\tau + S]/2}}{2(\lambda - e^{-[i\tau + S]/2})^2} \left[e^{-ih(R-S) + \delta[i\tau - R + S]} \left(\frac{1}{(i\tau - R + S)^2} - \frac{\delta + ih}{i\tau - R + S} \right) \right] \right] \right\}. \tag{82}
\end{aligned}$$

Our purpose, as in the proof of Lemma 9.3, is to examine the singularity of the kernel as $i\tau - R + S \rightarrow 0$. As in the analogous case of the sum for $j \geq 0$, some cancellation occurs.

Proceeding in a manner similar to our earlier calculation, we temporarily introduce the notation

$$x = i\tau - R + S, \quad \text{and} \quad \alpha = e^{-R/2},$$

so that

$$i\tau + S = x + R \quad \text{and} \quad e^{-[i\tau+S]/2} = \alpha e^{-x/2}.$$

As before we let $f(x) = (\alpha e^{-x/2} - \lambda)^{-1}$ but now $\Psi_-(x) = e^{2\delta x} f(x)$. Then we see that the sum of the terms containing negative powers of x equals $e^{-ih(R-S)+\delta x} Q_{-\delta}(x)$, where $Q_{-\delta}$ is defined in (73). Arguing as in the case $j \geq 0$, by Lemma 11.1 we obtain that the sum of the terms containing the negative powers of $i\tau - R + S$ on the righthand side of (82) equals

$$e^{-ih(R-S)+\delta x} Q_{-\delta}(i\tau - R + S) = \frac{e^{-ih(R-S)+\delta[i\tau-R+S]-R/2}}{2(e^{-[i\tau+S]/2} - \lambda)^2(e^{-R/2} - \lambda)^2} \psi_3^-(\tau, \lambda),$$

where we have set

$$\varphi_j^-(\tau) = G_j^-(i\tau - R + S) \quad \text{for } j = 1, \dots, 6, \quad (83)$$

and

$$\begin{aligned} \psi_3^-(\tau, \lambda) &= (e^{-[i\tau+S]/2} - \lambda)(e^{-R/2} - \lambda)\varphi_1^-(\tau) \\ &\quad + (e^{-R/2} - \lambda)^2\varphi_2^-(\tau) + (1 - e^{R/2}\lambda)(e^{[i\tau+S]/2} - \lambda)^2\varphi_3^-(\tau) \\ &\quad + e^{-R/2}\lambda\varphi_4^-(\tau) + \lambda\varphi_5^-(\tau) + e^{-R}\varphi_6^-(\tau). \end{aligned} \quad (84)$$

Therefore, using the argument above, we may simplify the expression (82) and obtain that

$$\sum_{j \leq -1} II^* = \frac{e^{-ih(R-S)+\delta[i\tau-R+S]-R/2}}{2(e^{-[i\tau+S]/2} - \lambda)^2(e^{-R/2} - \lambda)^2} \psi_3^-(\tau, \lambda).$$

Next, recalling that G_1^-, \dots, G_6^- are defined in (85), we have that

$$\begin{aligned} |\psi_3^-(\tau, \lambda)| &\leq C_M \left(|G_1^-(i\tau - R + S)| + |G_2^-(i\tau - R + S)| + e^{S+R/2} |G_3^-(i\tau - R + S)| \right. \\ &\quad \left. + |G_4^-(i\tau - R + S)| + |G_5^-(i\tau - R + S)| + e^{-R} |G_6^-(i\tau - R + S)| \right) \\ &\leq C_M (e^{S/2+2\delta R} + e^{S+R(2\delta+1/2)} + e^S + e^{2R}) \\ &\leq C_M (e^{S+R(2\delta+1/2)} + e^{2R}). \end{aligned}$$

This proves Lemma 9.5. □

Finally we prove Lemma 11.1.

Proof of Lemma 11.1. For simplicity of notation, we provide the details for $Q_{+\delta}$ only. We write F in place of Ψ_{\pm} .

We begin by noticing that

$$\begin{aligned} 2 \frac{F(x) - F(0)}{x} &= \frac{2}{x} ((e^{-2\delta x} - 1)f(x) + f(x) - f(0)) \\ &= \frac{e^{-2\delta x} - 1}{x/2} f(x) + 2 \frac{f(x) - f(0)}{x}, \end{aligned}$$

and that

$$\begin{aligned} F'(x) + F'(0) &= e^{-2\delta x} \left(f'(x) - 2\delta f(x) \right) + f'(0) - 2\delta f(0) \\ &= (e^{-2\delta x} - 1)f'(x) + (f'(x) + f'(0)) - 2\delta(e^{-2\delta x} f(x) + f(0)). \end{aligned}$$

Therefore

$$\begin{aligned} &2 \frac{F(x) - F(0)}{x} - (F'(x) + F'(0)) \\ &= 2 \frac{f(x) - f(0)}{x} - (f'(x) - f'(0)) + \frac{e^{-2\delta x} - 1}{x/2} f(x) - (e^{-2\delta x} - 1)f'(x) \\ &\quad + 2\delta(e^{-2\delta x} f(x) + f(0)) \\ &= 2 \frac{f(x) - f(0)}{x} - (f'(x) - f'(0)) + \left(\frac{e^{-2\delta x} - 1}{x/2} + 2\delta e^{-2\delta x} + 2\delta \right) f(x) \\ &\quad - (e^{-2\delta x} - 1)f'(x) - 2\delta(f(x) - f(0)) \\ &= 2 \frac{f(x) - f(0)}{x} - (f'(x) - f'(0)) + \left(\frac{e^{-2\delta x} - 1}{x/2} + 2\delta e^{-2\delta x} + 2\delta \right) f(x) \\ &\quad - (e^{-2\delta x} - 1 + 2\delta x)f'(x) - 2\delta x \left(\frac{f(x) - f(0)}{x} - f'(x) \right) \\ &\equiv A_1 + A_2 + A_3 + A_4. \end{aligned}$$

While A_2 and A_3 have simple expressions, we need to manipulate A_1 and A_4 to obtain some simplifications. Setting $\Psi_1(x) = 2 \frac{1 - e^{-x/2}}{x/2} - e^{-x/2} - 1$, we have that

$$\begin{aligned} A_1 &= \frac{\alpha}{(\alpha e^{-x/2} - \lambda)(\alpha - \lambda)} \left(\frac{1 - e^{-x/2}}{x/2} \right) - \left(\frac{\alpha e^{-x/2}}{2(\alpha e^{-x/2} - \lambda)^2} + \frac{\alpha}{2(\alpha - \lambda)^2} \right) \\ &= \frac{\alpha}{2(\alpha e^{-x/2} - \lambda)(\alpha - \lambda)} \left[2 \frac{1 - e^{-x/2}}{x/2} - \frac{e^{-x/2}(\alpha - \lambda)}{\alpha e^{-x/2} - \lambda} + \frac{\alpha e^{-x/2} - \lambda}{\alpha - \lambda} \right] \\ &= \frac{\alpha}{2(\alpha e^{-x/2} - \lambda)(\alpha - \lambda)} \left[\Psi_1(x) - e^{-x/2} \left(\frac{\alpha - \lambda}{\alpha e^{-x/2} - \lambda} - 1 \right) - \left(\frac{\alpha e^{-x/2} - \lambda}{\alpha - \lambda} - 1 \right) \right] \\ &= \frac{\alpha}{2(\alpha e^{-x/2} - \lambda)^2(\alpha - \lambda)^2} \left[\Psi_1(x)(\alpha e^{-x/2} - \lambda)(\alpha - \lambda) - \alpha\lambda(1 - e^{-x/2})^2 \right]. \end{aligned}$$

Next,

$$\begin{aligned}
A_4 &= -2\delta x \left[\frac{\alpha}{2(\alpha e^{-x/2} - \lambda)(\alpha - \lambda)} \left(\frac{1 - e^{-x/2}}{x/2} \right) - \frac{\alpha e^{-x/2}}{2(\alpha e^{-x/2} - \lambda)^2} \right] \\
&= -\frac{\alpha \delta x}{(\alpha e^{-x/2} - \lambda)} \left[\left(\frac{1 - e^{-x/2}}{x/2} - 1 \right) \frac{1}{\alpha - \lambda} + \frac{\alpha(e^{-x/2} - 1)}{(\alpha - \lambda)(\alpha e^{-x/2} - \lambda)} - \frac{e^{-x/2} - 1}{\alpha e^{-x/2} - \lambda} \right] \\
&= -\frac{\alpha \delta x}{(\alpha e^{-x/2} - \lambda)^2(\alpha - \lambda)} \left[\left(\frac{1 - e^{-x/2}}{x/2} - 1 \right) (\alpha e^{-x/2} - \lambda) + (e^{-x/2} - 1)\lambda \right].
\end{aligned}$$

Now we turn to the second summand in (73). It holds that

$$\begin{aligned}
F(x) - F(0) &= (e^{-2\delta x} - 1) \frac{1}{\alpha e^{-x/2} - \lambda} + \frac{\alpha}{(\alpha e^{-x/2} - \lambda)(\alpha - \lambda)} (1 - e^{-x/2}) \\
&= \frac{1}{(\alpha e^{-x/2} - \lambda)(\alpha - \lambda)} \left[(e^{-2\delta x} - 1)(\alpha - \lambda) + \alpha(1 - e^{-x/2}) \right]
\end{aligned}$$

and

$$\begin{aligned}
e^{-2\delta x} f'(x) - f'(0) &= (e^{-2\delta x} - 1) \frac{\alpha e^{-x/2}}{2(\alpha e^{-x/2} - \lambda)^2} + \frac{\alpha(e^{-x/2} - 1)}{2(\alpha e^{-x/2} - \lambda)^2} \\
&\quad + \frac{\alpha^2(e^{-x/2} - 1)(\alpha(e^{-x/2} + 1) + 2\lambda)}{2(\alpha e^{-x/2} - \lambda)^2(\alpha - \lambda)^2} \\
&= \frac{\alpha}{2(\alpha e^{-x/2} - \lambda)^2(\alpha - \lambda)^2} \left[e^{-x/2}(e^{-2\delta x} - 1)(\alpha - \lambda)^2 \right. \\
&\quad \left. + (e^{-x/2} - 1)(\alpha - \lambda)^2 + \alpha(1 - e^{x/2})(\alpha(e^{-x/2} + 1) + 2\lambda) \right].
\end{aligned}$$

Therefore

$$\begin{aligned}
Q_\delta(x) &= \frac{\alpha}{2(\alpha e^{-x/2} - \lambda)^2(\alpha - \lambda)^2} \left\{ \frac{ihx - 1}{x^2} \left[\Psi_1(x)(\alpha e^{-x/2} - \lambda)(\alpha - \lambda) - \alpha\lambda(1 - e^{-x/2})^2 \right. \right. \\
&\quad + 2 \left(\frac{e^{-2\delta x} - 1}{x/2} + 2\delta e^{-2\delta x} + 2\delta \right) \frac{(\alpha - \lambda)^2(\alpha e^{-x/2} - \lambda)}{\alpha} - (e^{-2\delta x} - 1 + 2\delta x)e^{-x/2}(\alpha - \lambda)^2 \\
&\quad \left. \left. - 2\delta x \left(\frac{1 - e^{-x/2}}{x/2} - 1 \right) (\alpha e^{-x/2} - \lambda)(\alpha - \lambda) + \delta x(e^{-x/2} - 1)\lambda(\alpha - \lambda) \right] \right. \\
&\quad + \frac{1}{x} \left[2(h^2 - \delta^2) \left((e^{-2\delta x} - 1) \frac{(\alpha - \lambda)^2(\alpha e^{-x/2} - \lambda)}{\alpha} + (1 - e^{-x/2})(\alpha - \lambda)(\alpha e^{-x/2} - \lambda) \right) \right. \\
&\quad \left. \left. + \delta \left(e^{-x/2}(e^{-2\delta x} - 1)(\alpha - \lambda)^2 + (e^{-x/2} - 1)(\alpha - \lambda)^2 + \alpha(1 - e^{-x/2})(\alpha(e^{-x/2} + 1) + 2\lambda) \right) \right] \right\} \\
&= \frac{\alpha}{2(\alpha e^{-x/2} - \lambda)^2(\alpha - \lambda)^2} \left[(\alpha e^{-x/2} - \lambda)(\alpha - \lambda)G_1(x) + (\alpha - \lambda)^2G_2(x) \right. \\
&\quad \left. + \frac{(\alpha - \lambda)^2(\alpha e^{-x/2} - \lambda)}{\alpha}G_3(x) + \alpha\lambda G_4(x) + \lambda(\alpha - \lambda)G_5(x) + \alpha^2G_6(x) \right],
\end{aligned}$$

where G_1, \dots, G_6 are entire functions of x , independent of α and λ . Explicitly, recalling the dependence on the choice of the sign in front of δ , they are given by

$$\begin{aligned}
G_1^\pm(x) &= \frac{ihx - 1}{x^2} \left(2 \frac{1 - e^{-x/2}}{x/2} - e^{-x/2} - 1 \right) - 2\delta x \left(\frac{1 - e^{-x/2}}{x/2} - 1 \right) + (h^2 - \delta^2) \frac{1 - e^{-x/2}}{x/2} \\
G_2^\pm(x) &= \frac{ihx - 1}{x^2} \left[-e^{-x/2}(e^{\mp 2\delta x} - 1 \pm 2\delta x) \right] \pm \delta e^{-x/2} \left(\frac{e^{\mp 2\delta x} - 1}{x/2} + \frac{e^{-x/2} - 1}{x} \right) \\
G_3^\pm(x) &= 2 \frac{ihx - 1}{x^2} \left(\frac{e^{\mp 2\delta x} - 1}{x/2} \pm 2\delta e^{-2\delta x} \pm 2\delta \right) + (h^2 - \delta^2) \frac{e^{\mp 2\delta x} - 1}{x/2} \\
G_4^\pm(x) &= -\frac{ihx - 1}{x^2} (1 - e^{-x/2})^2 \pm 2\delta e^{-x/2} \frac{1 - e^{-x/2}}{x} \\
G_5^\pm(x) &= \mp 2\delta \frac{ihx - 1}{x^2} (e^{-x/2} - 1) \\
G_6^\pm(x) &= \pm \delta \frac{e^{-x} - 1}{x}.
\end{aligned}$$

(85)

That completes the proof of Lemma 11.1. \square

12. CONCLUDING REMARKS

We have provided a rather complete analysis of the Bergman kernel for the non-smooth worms D_β and D'_β . Our work has been facilitated by the fact that the boundaries of these domains are Levi flat, hence they are qualitatively like product domains (and our resulting formulas reflect this structure). Another way to look at this is that the domains D_β and D'_β possess many more symmetries with respect to the smooth worm \mathcal{W}_β , since the translations in the real part of z_1 , $(z_1, z_2) \rightarrow (z_1 + a, z_2)$ with $a \in \mathbb{R}$, are automorphisms of both D_β and D'_β .

It is a matter of considerable interest to perform the analogous analysis on the smooth worm \mathcal{W}_β . That work will be of a different nature, for the boundary of \mathcal{W}_β is strongly pseudoconvex at all points except those on a singular annulus in the boundary. There certainly is no “product structure”. We hope to complete such an analysis in a future paper.

The results that we present provide a new way to view the negative results of Kiselman, Barrett, and Christ about mapping properties of the Bergman projection on worm domains. We hope that they will serve as a stepping stone to future studies.

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